1) Solve the equation \((1234)x = (4321)\) in \(S_4\).

2a) Determine all zero divisors in \(Z_{11}\) and \(Z_{12}\).

2b) Determine all invertible elements in \(Z_{11}\) and \(Z_{12}\).

3) There are nine monic polynomials of degree 2 with coefficients in \(Z_5\). Determine for each of these if the polynomial is irreducible or not.

4) An isomorphism of a group onto itself is called an automorphism.
   a) Prove that the set \(Aut(G)\) of all automorphisms of a group \(G\) is itself a group with respect to composition.
   b) If \(g \in G\), let \(\pi_g(a) = g a g^{-1}\) for each \(a \in G\). Verify that the map \(\pi_g: G \rightarrow G\) belongs to \(Aut(G)\).
   c) Show that the map \(\pi: G \rightarrow Aut(G)\) which sends \(g\) to \(\pi_g\) is a homomorphism.

5) Let \(N\) be a normal subgroup of \(G\) and \(G/N\) denote the set of all right cosets of \(N\) in \(G\).

For \(Na \in G/N\) and \(Nb \in G/N\), let \((Na)(Nb) = N(ab)\). Show that this gives a well defined binary operation on \(G/N\) which satisfies all group axioms.

6a) Prove that the kernel of a ring homomorphism \(\theta: R \rightarrow S\) is an ideal of \(R\).

b) Show that \(\theta\) is injective if and only if \(\text{Ker } \theta = \{0\}\).

All claims that are made must be motivated. The exam will be corrected within two weeks.
1) \((1234) \Rightarrow x = (4321) \Rightarrow x = (1234)^3(1432) \Rightarrow x = 1432 \Rightarrow x = (13)(24)\)

2) \(\mathbb{Z}/11\) is a field as 11 is a prime. There are thus no zero divisors and all non-zero elements are invertible.

The classes of 2, 3, 4, 6, 8, 9 and 10 are zero divisors in \(\mathbb{Z}/12\) as \(2 \times 6, 3 \times 4, 6 \times 10\) and \(8 \times 9\) are all divisible by 12. The classes of 1, 5, 7, 11 are invertible in \(\mathbb{Z}/12\) as the squares of 1, 5, 7, 11 are all \(1 \mod 12\).

These lists are complete as no element \(a\) in a commutative ring \(A\) can be both invertible and a zero divisor. Indeed, if \(ab = 1\) and \(ac = 0\), then \(c = abc = acb = 0\). (Also, 0 is never invertible and never a zero divisor by definition.)

3) A monic polynomial \(p\) of degree 2 over a field is irreducible if and only if it is not a product of two monic linear polynomials over that field. If we write \(0, 1, 2\) also for their classes in \(\mathbb{Z}/3\), then

\[
\begin{align*}
x^2, x^2+x &= x(x+1), \quad x^2+2x &= x(x+2), \quad x^2+2x+1=(x+1)^2, \quad x^2+2=(x+1)(x+2), \quad x^2+x+1=(x+2)^2
\end{align*}
\]

is a complete list of all products of two linear monic polynomials over \(\mathbb{Z}/3\). These are thus the reducible monic quadratic polynomials over \(\mathbb{Z}/3\). The other three monic quadratic polynomials over \(\mathbb{Z}/3\) : \(x^2+1, x^2+x+2, x^2+2x+2\) are therefore irreducible.

4a) \(\text{Aut}(G)\) is a subset of the group \(\text{Sym}(G)\) of all permutations of \(G\). One may thus use the criterion to check that if \(\text{Aut}(G)\) is a subgroup of \(\text{Sym}(G)\). But \(\text{Aut}(G)\) is non-empty as it contains the identity map \(id: G \rightarrow G\). Moreover, as the composition of two automorphisms is an automorphism just like the inverse of an automorphism, we conclude that \(\text{Aut}(G)\) is a subgroup of \(\text{Sym}(G)\).

4b) \(\pi_G: G \rightarrow G\) is bijective with inverse \(\pi_{G^1}\) as \(\pi_G(\pi_G(a)) = g'g = a\). It is also a homomorphism as \(\pi_G(ab) = (gag^{-1})(gbg^{-1}) = \pi_G(a) \pi_G(b)\). Hence \(\pi_G \in \text{Aut}(G)\).

4c) \(\pi_G(a) = gh(a)(gh)^{-1} = g(hah^{-1})g^{-1} = \pi_G(ha^{-1})\) for all \(a \in G\). Hence \(\pi_G = \pi_G \pi_G\) for all \(g, h \in G\), as was to be proved.

5) See Durbin's book and the lecture notes.

6) See Durbin's book and the lecture notes.