MATHEMATICS University of Gothenburg and Chalmers University of Technology
Examination in algebra (MMG 500 and MVE 150), 2012-08-25 8:30 – 12:30
No books, written notes or any other aids are allowed.
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1) Solve the equation $(1234)\,x = (4321)$ in S_4. 3p

2a) Determine all zero divisors in $\mathbf{Z}_{11}$ and $\mathbf{Z}_{12}$. 4p

2b) Determine all invertible elements in $\mathbf{Z}_{11}$ and $\mathbf{Z}_{12}$.

3) There are nine monic polynomials of degree 2 with coefficients in $\mathbf{Z}_3$. Determine for each of these if the polynomial is irreducible or not. 4p

4) An isomorphism of a group onto itself is called an automorphism. 6p

a) Prove that the set $\mathrm{Aut}(G)$ of all automorphisms of a group G is itself a group with respect to composition.

b) If $g \in G$, let $\pi_g(a) = gag^{-1}$ for each $a \in G$. Verify that the map $\pi_g\colon G \to G$ belongs to $\mathrm{Aut}(G)$.

c) Show that the map $\pi\colon G \to \mathrm{Aut}(G)$ which sends g to π_g is a homomorphism.

5) Let N be a normal subgroup of G and G/N denote the set of all right cosets of N in G. 4p

For $Na \in G/N$ and $Nb \in G/N$, let $(Na)(Nb) = N(ab)$. Show that this gives a well defined binary operation on G/N which satisfies all group axioms.

6a) Prove that the kernel of a ring homomorphism $\theta\colon R \to S$ is an ideal of R. 4p

b) Show that θ is injective if and only if $\mathrm{Ker}\,\theta = \{0\}$.

All claims that are made must be motivated. The exam will be corrected within two weeks.

Solutions Algebra MMG 500/MVE 150 August 2012

1) $(1234)\,x = (4321) \Leftrightarrow x = (1234)^{-1}(1432) \Leftrightarrow x = (1432)\,(1432) \Leftrightarrow x = (13)(24)$

2) $\mathbf{Z}_{11}$ is a field as 11 is a prime. There are thus no zero divisors and all non-zero elements are invertible.
The classes of 2,3,4,6,8,9 and 10 are zero divisors in $\mathbf{Z}_{12}$ as 2×6, 3×4, 6×10 and 8×9 are all divisible by 12. The classes of 1,5,7,11 are invertible in $\mathbf{Z}_{12}$ as the squares of 1,5,7,11 are all $\equiv 1 \pmod{12}$. These lists are complete as no element a in a commutative ring A can be both invertible and a zero divisor. Indeed, if $ab=1$ and $ac=0$, then $c=abc=acb=0$. (Also, 0 is never invertible and never a zero divisor by definition.)

3) A monic polynomial p of degree 2 over a field is irreducible if and only if it is not a product of two monic linear polynomials over that field. If we write 0,1,2 also for their classes in $\mathbf{Z}_3$, then

$$x^2,\ x^2+x = x(x+1),\ x^2+2x = x(x+2)\ ,\ x^2+2x+1=(x+1)^2,\ x^2+2=(x+1)(x+2)\ ,\ x^2+x+1=(x+2)^2$$

is a complete list of all products of two linear monic polynomials over $\mathbf{Z}_3$. These are thus the reducible monic quadratic polynomials over $\mathbf{Z}_3$. The other three monic quadratic polynomials over $\mathbf{Z}_3$: x^2+1, x^2+x+2, x^2+2x+2 are therefore irreducible.

4a) Aut (G) is a subset of the group Sym(G) of all permutations of G. One may thus use the criterion to check that if Aut (G) is a subgroup of Sym(G). But Aut (G) is non-empty as it contains the identity map $id\colon G \to G$. Moreover, as the composition of two automorphisms is an automorphism just like the inverse of an automorphism, we conclude that Aut (G) is a subgroup of Sym(G).

4b) $\pi_g : G \to G$ is bijective with inverse $\pi_{g^{-1}}$ as $\pi_{g^{-1}}(\pi_g(a)) = g^{-1}(gag^{-1})g = a$. It is also a homomorphism as $\pi_g(ab) = g(ab)g^{-1} = (gag^{-1})(gbg^{-1}) = \pi_g(a)\,\pi_g(b)$. Hence $\pi_g \in \text{Aut } G$.

4c) $\pi_{gh}(a) = gh(a)(gh)^{-1} = g(hah^{-1})g^{-1} = \pi_g(hah^{-1}) = \pi_g(\pi_h(a))$ for all $a \in G$. Hence $\pi_{gh} = \pi_g\pi_h$ for all $g,h \in G$, as was to be proved.

5) See Durbin's book and the lecture notes.

6) See Durbin's book and the lecture notes.