MATHEMATICS University of Gothenburg and Chalmers University of Technology
Examination in algebra (MMG 500 and MVE 150), 2012-06-07
No books, written notes or any other aids are allowed.
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1a) Define what is meant by a group homomorphism. 1p

b) Let **C*** be the multiplicative group of all complex numbers different from zero and $\varepsilon \in \mathbf{C}^*$. Show that the map θ: $\mathbf{Z} \to \mathbf{C}^*$, which sends k to ε^k is a group homomorphism. 1p

c) Show that the kernel of θ is of finite index in **Z** if and only if $\varepsilon = e^{2\pi i q}$ for some rational number q. 2p

2. Prove or disprove the following isomorphism statements. 4p

a) $\mathbf{Z}_{2012} \approx \mathbf{Z}_4 \times \mathbf{Z}_{503}$

b) $\mathbf{Z}_2 \times \mathbf{Z}_8 \approx \mathbf{Z}_4 \times \mathbf{Z}_4$

c) $\mathbf{Z}_{14} \times \mathbf{Z}_{15} \approx \mathbf{Z}_{10} \times \mathbf{Z}_{21}$

3. A tetrahedron is said to be regular if all its four faces are equilateral triangles.

a) Show that the symmetry group G of a regular tetrahedron acts transitively on its set of four vertices. 1p

b) Show that the G is isomorphic to the alternating group A_4. 2p

4. Let $R = \mathbf{C}[x, y]$ be the ring of all polynomials in x and y over **C**.

a) Show that the set of all polynomials $f(x,y)$ with $f(0, 0)=0$ form an ideal I of R. 1p

b) Prove that I is not a principal ideal of R. 3p

5. Formulate and prove Lagrange's theorem for groups. 4p

6. Prove that a polynomial of degree n over **C** has at most n zeros. 3p

7a) Define what is meant by a Euclidean domain. 1p

b) Prove that the ring of integers $\mathbf{Z}[i]$ is a Euclidean domain. 2p

All claims that are made must be motivated. The exam will be corrected within two weeks.

Solutions to exam in algebra (MMG 500 + MVE 150) **2012-06-07**

1a) See Durbin's book for the definition of a group homomorphism.

b) Let $m, n \in \mathbf{Z}$. Then $\theta(m+n) = \varepsilon^{m+n} = \varepsilon^m \varepsilon^n = \theta(m)\,\theta(n)$.

c) The index $[\mathbf{Z}:\ker\theta]$ is equal to the order of the quotient group $\mathbf{Z}/\ker\theta$. By the fundamental homomorphism theorem for groups we have further that $\mathbf{Z}/\ker\theta \approx \operatorname{im}\theta = \langle\varepsilon\rangle$. Hence $[\mathbf{Z}:\ker\theta] = |\langle\varepsilon\rangle| = o(\varepsilon)$. Now choose $z = x+iy \in \mathbf{C}$ with $\varepsilon = e^z$. Then, $\varepsilon^k = e^{kz} = e^{kx}(\cos ky + i\sin ky) = 1$ if and only if $x=0$ and $ky \in \{2\pi n : n \in \mathbf{Z}\}$. Thus, ε is of finite order in $\mathbf{C}^*$ if and only if $\varepsilon = e^{2\pi i q}$ for some rational number q, thereby proving the assertion.

2. If m,n are relatively prime, then it is shown in the lecture notes that $\mathbf{Z}_{mn} \approx \mathbf{Z}_m \times \mathbf{Z}_n$. In particular, $\mathbf{Z}_{2012} \approx \mathbf{Z}_4 \times \mathbf{Z}_{503}$, $\mathbf{Z}_{210} \approx \mathbf{Z}_{14} \times \mathbf{Z}_{15}$ and $\mathbf{Z}_{210} \approx \mathbf{Z}_{10} \times \mathbf{Z}_{21}$ such that (a) and (c) hold. But (b) cannot hold as $\mathbf{Z}_2 \times \mathbf{Z}_8$ contains elements of order 8 like $([1]_2, [1]_8)$, while all elements in $\mathbf{Z}_4 \times \mathbf{Z}_4$ are of order 1,2 or 4.

3a). If P and Q are two vertices, let R be another vertex. We may then rotate the tetrahedron while keeping R fixed such that P goes to Q. Hence the action is transitive.

b) Let S be the set of the four vertices. Then any symmetry of the tetrahedron is uniquely determined by its action on S. If we number the vertices 1,2,3,4, there is thus an injective homomorphism from G to $\mathrm{Sym}(S) = S_4$. The elements in the stabilizer G_i under the action of G on S will correspond to the 3 rotations of the opposite triangle. As $\mathrm{Orb}(i) = S$, we have thus by a theorem in Durbin's book that $o(G) = |\mathrm{Orb}(i)|\,o(G_i) = 4\times 3 = 12$. The action of G on S gives rise to even permutations of S. Indeed, the eight rotations with exactly one fixed vertex correspond to the 3-cycles in S_4 and the three reflections of the tetrahedron correspond to products of two disjoint transpositions in S_4. Hence the homomorphism from G to $\mathrm{Sym}(S)$ induce an isomorphism from G to the alternating group A_4.

4a). The evaluation map which sends $f(x,y)$ to $f(0,0)$ is a homomorphism of rings from $R = \mathbf{C}[x,y]$ to $\mathbf{C}$. Its kernel I is thus an ideal by a theorem in Durbin's book.

b) If I were a principal ideal generated by $f(x,y) \in I$, then as $x, y \in I$ we would have other polynomials $g(x,y), h(x,y) \in R$ with $f(x,y)g(x,y) = x$ and $f(x,y)h(x,y) = y$. But this is only possible if $f(x,y)$ is a non-zero constant in $\mathbf{C}$, thereby contradicting the assumption that $f(x,y) \in I$. Hence I cannot be a principal ideal.