Foundations of Probability Theory (MVE140 – MSA150)

Friday 9th of April 2021 resit examination questions

This examination has five problems with a maximum of 20 credit points for a fully satisfactory solution, so the maximal total is 100 credit points. To pass the course, you need to score at least 40 points (50 if you are a PhD student). You should keep a zoom session with a camera on showing you working for the whole duration of the exam. A recording will be made which will be deleted soon after the results are released. A member of staff is available for consultation on zoom around 3pm and 5pm.

Examination Questions

1. Let $[X, \mathcal{X}]$ be a measurable space and let $A \subset X,\ A \in \mathcal{X}$ be a fixed measurable set. Define the system of subsets $\mathcal{X}_A = \{A \cap B :\ B \in \mathcal{X}\}$.

   a) Show that $\mathcal{X}_A$ is a σ-field.

   b) Which of $[X, \mathcal{X}]$-measurable functions $X \mapsto \mathbb{R}$ are also $[X, \mathcal{X}_A]$-measurable?

2. An organiser of a TV show suggests a participant to choose between two urns and draw one ball from it. If the participant draws a white ball he or she wins a prise, but if the ball is black, the participant quits the show. There are altogether 4 balls: 2 white and 2 black. How the organiser should in advance distribute the balls over the two urns (not all four in the same urn) so that to minimise the chance of a participant to win?

3. Let $S_n = \sum_{i=1}^n \xi_i$ for independent identically distributed non-negative random variables $\xi_1, \xi_2, \dots$. Show that for every $c > 0$ there is an $\alpha = \alpha(c) > 0$ such that $\mathbf{P}\{S_n \le c\} \le \exp\{-n\alpha\}$.

4. The joint density of the vector (ξ, η) is given by $f(x, y) = a(x + 2y)$ in the square $[0, 1]^2$ and 0 outside of it. Find the constant a and:

   a. The joint c.d.f.;

b. The marginal p.d.f. of ξ and η;

c. The probability that $\eta < \xi^2$;

d. Conditional distribution of ξ given $\eta = 1/2$ and the conditional expectation $\mathbf{E}[\xi \mid \eta]$.

5. Let ξ_n be Geometrically distributed with parameters p_n random variables (i.e. $\mathbf{P}\{\xi_n = k\} = p_n(1-p_n)^{k-1},\ k = 1, 2, \dots$), where $\lim_{n\to\infty} p_n = 0$. Show that the sequence $\{p_n\xi_n\}$ converges weakly to an exponentially distributed random variable ξ with parameter 1 (i.e. $\mathbf{P}\{\xi \le x\} = 1 - e^{-x},\ x \ge 0$).