Foundations of Probability Theory (MVE140 – MSA150)

Wednesday 16th of January 2019 examination questions

You are allowed to use a dictionary (to and from English) and up to a maximum of one double-sided page of your own handwritten notes. This examination has five problems with a maximum of 20 credit points for a fully satisfactory solution, so the maximal total is 100 credit points. To pass the course, you need to score at least 40 points (50 if you are a PhD student). A member of staff is available at the examination site around 10:30am and noon.

Examination Questions

1. Let $\mathcal{F}$ be the system of segments $\{[0, 2^{-n}),\ n \in \mathbb{N}\}$ in $\mathbb{R}$.

   (a) Is $\mathcal{F}$ a σ-field?

   (b) Describe the σ-filed $\sigma(\mathcal{F})$ it generates.

   (c) Which functions from $\mathbb{R}$ to $\mathbb{R}$ are $\sigma(\mathcal{F})$-measurable?

   Solution.

   (a) No, because, e.g., $[0, 1/4) \cup [0, 1/2]^{\mathsf{c}}$ is not of the type $[0, 2^{-n})$.

   (b) $\sigma(\mathcal{F})$ consists of $(-\infty, 0), [1/2, +\infty)$, the semi-intervals of the type $[2^{-m-1}, 2^{-n})$ and their countable unions.

   (c) Measurable functions are the ones which are constants on the sets from $\sigma(\mathcal{F})$ above (in particular, they are right-continuous).

2. We toss a coin which shows a Head with probability p until two consecutive Heads or Tails appear. What is the probability that an even number of tosses will be required?

Solution. If HH appear at the tosses number $2k-1$ and $2k$, then the previous tosses must be alternating Head-Tail ending with Tail at toss number $2k-2$. The probability of this is $p^2(p(1-p))^{k-1}$. Similarly, the probability of Tail-Tail first observed at tosses $2k-1$ and $2k$: $(1-p)^2(p(1-p))^{k-1}$. Altogether,

$$(p^2+(1-p)^2)\sum_{k=1}^{\infty}(p(1-p))^{k-1} = \frac{p^2+(1-p)^2}{1-p(1-p)}.$$

3. The joint density of the random vector (ξ,η) is given by the density $f_{(\xi,\eta)}(x,y) = ye^{-y(x+1)}$, $x,y>0$. Find
   (a) the marginal distributions of ξ and η;
   (b) the expectation and variance of ξ and η;
   (c) The conditional c.d.f. $F_{\xi|\eta}(x|y)$;
   (d) The conditional expectation $\mathbf{E}[\xi \mid \eta]$ and its distribution.

   Solution.
   (a) By integration, $f_\xi(x) = \int ye^{-y(x+1)}dy = 1/(1+x)^2$. Similarly, $f_\eta(y) = e^{-y}$ which is Exp(1)-distribution.
   (b) $\mathbf{E}\,\xi = \int_0^\infty x/(1+x)^2dx = +\infty = \mathbf{var}\,\xi$, $\mathbf{E}\,\eta = 1 = \mathbf{var}\,\eta$.
   (c) $f_{\xi|\eta}(x|y) = \frac{ye^{-y(x+1)}}{e^{-y}} = ye^{-yx}$, hence $F_{\xi|\eta}(x|y) = 1-e^{-yx}$, i.e. Exp($y$)-distribution. Thus the mean is:
   (d) $\zeta = \mathbf{E}[\xi \mid \eta] = 1/\eta$. Its p.d.f. is then $F_\zeta(z) = \mathbf{P}\{1/\eta \le z\} = \mathbf{P}\{\eta > 1/z\} = e^{-1/z}\,\mathbb{1}\{z>0\}$.

4. Show that if a sequence of random variables ξ_n on the same probability space converges to ξ in distribution (weakly) and a sequence of random variables η_n on the same space converges to a constant c in probability, then $\eta_n\xi_n \to c\xi$ in distribution.

Solution. For any $\varepsilon > 0$ and a real x,

$$F_n(x) \stackrel{\text{def}}{=} \mathbf{P}\{\eta_n\xi_n \le x\} = \mathbf{P}\{\eta_n\xi_n \le x;\ |\eta_n - c| \le \varepsilon\} + \mathbf{P}\{\eta_n\xi_n \le x;\ |\eta_n - c| > \varepsilon\}$$
$$\le \mathbf{P}\{\xi_n \le \frac{x}{c-\varepsilon}\} + \mathbf{P}\{|\eta_n - c| > \varepsilon\}.$$

On the other hand,

$$\mathbf{P}\{\eta_n\xi_n \le x\} \ge \mathbf{P}\{\eta_n\xi_n \le x;\ |\eta_n - c| \le \varepsilon\}$$
$$\ge \mathbf{P}\{(c+\varepsilon)\xi_n \le x;\ |\eta_n - c| \le \varepsilon\} = \mathbf{P}\{\xi_n \le \frac{x}{c+\varepsilon}\} - \mathbf{P}\{\xi_n \le \frac{x}{c+\varepsilon};\ |\eta_n - c| > \varepsilon\}.$$

Since $\mathbf{P}\{|\eta_n - c| > \varepsilon\} \to 0$, we get

$$\lim_{n\to\infty} \mathbf{P}\{\xi_n \le \frac{x}{c+\varepsilon}\} \le \liminf F_n(x) \le \limsup F_n(x) \le \lim_{n\to\infty} \mathbf{P}\{\xi_n \le \frac{x}{c-\varepsilon}\}.$$

If F_ξ is continuous at x/c, i.e. $F_{c\xi}$ is continuous at x, it is also continuous in its sufficiently small neighbourhood, thus, using that $\xi_n \to \xi$ in distribution, we have that for all small enough ε,

$$\lim_{n\to\infty} \mathbf{P}\{\xi_n \le \frac{x}{c \pm \varepsilon}\} = \mathbf{P}\{\xi \le \frac{x}{c \pm \varepsilon}\}.$$

Now take the limit $\varepsilon \downarrow 0$ and get $F_n(x) \to F_\xi(x/c) = F_{c\xi}(x)$, that is required.

5. Let ξ_i, $i = 1, 2, \ldots$ be non-negative independent identically distributed random variables with the mean m and variance σ^2. Find the weak limit of $\sqrt{\sum_{i=1}^n \xi_i} - \sqrt{nm}$.
Hint: Use the Central Limit theorem and the fact formulated in the previous question (even if you have not proved it).

Solution. Let $S_n = \sum_{i=1}^{n} \xi_i$. Then

$$\sqrt{S_n} - \sqrt{nm} = \frac{S_n - nm}{\sqrt{S_n} + \sqrt{mn}} = \frac{S_n - nm}{\sqrt{n}\sigma} \frac{\sigma}{\sqrt{S_n/n} + \sqrt{m}}.$$

By the CLT, the first fraction weakly converges to the standard Normal law $\mathcal{N}(0,1)$. The second, by the Law of Large Numbers converges in probability to $\sigma/(2\sqrt{m})$. Thus, by the previous question, the weak limiting distribution is Normal $\mathcal{N}(0, \sigma^2/(4m))$.