Foundations of Probability Theory (MVE140 – MSA150)

Wednesday 10th of January 2018 examination questions

You are allowed to use a dictionary (to and from English) and up to a maximum of 2 double-sided pages of your own written notes. This examination has five problems with a maximum of 20 credit points for a fully satisfactory solution, so the maximal total is 100 credit points. To pass the course, you need to score at least 40 points. A member of staff is available at the examination site around 10:30am and 12pm.

Examination Questions

1. Let $\{\mathcal{B}_\alpha : \alpha \in I\}$ be an arbitrary family of σ-fields of subsets of Ω. Show that $\cap_{\alpha\in I}\mathcal{B}_\alpha$ is a σ-field.

   Solution. Since $\Omega \in \mathcal{B}_\alpha$ for every α, then $\Omega \in \cap_{\alpha\in I}\mathcal{B}_\alpha$. Next, if $B \in \cap_\alpha \mathcal{B}_\alpha$ then $B \in \mathcal{B}_\alpha$ for all α, hence $B^{\mathrm{c}} \in \mathcal{B}_\alpha$ for all α, i.e. $B^{\mathrm{c}} \in \cap_{\alpha\in I}\mathcal{B}_\alpha$. Similarly it is shown that the countable unions are in $\cap_{\alpha\in I}\mathcal{B}_\alpha$.

2. Let $\xi_1, \xi_2, \ldots$ be a sequence of independent Bernoulli-distributed with parameter p random variables and ν be a Poisson distributed random variable with parameter λ independent of $\xi_n's$. Find the characteristic function of the sum $\zeta = \sum_{n=1}^{\nu} \xi_n$ (the sum is understood as 0, if $\nu = 0$). Find the expectation and the variance of ζ.

   Solution. $\varphi_\xi(t) = 1 - p + pe^{it}$,

$$\varphi_\zeta(t) = \sum_{n=0}^{\infty} (\mathbf{E}\, e^{it\xi})^n \lambda^n/(n!)\, e^{-\lambda} = \sum_{n=0}^{\infty} (\varphi_\xi(t)\lambda)^n/(n!)\, e^{-\lambda} = e^{\lambda(\varphi_\xi(t)-1)}$$

$$= e^{\lambda p(e^{it}-1)} = 1 + it\lambda p - \frac{t^2}{2}(\lambda p + \lambda^2 p^2) + o(t^2)\,.$$

   Hence $\mathbf{E}\,\zeta = \lambda p$, $\mathbf{E}\,\zeta^2 = \lambda p + \lambda^2 p^2$ and $\mathbf{var}\,\zeta = \lambda p$ also.

3. A piece of wire is cut into two pieces at an arbitrary point. One piece is bent into a square, the other piece – into a circle. Find the probability that the area of the square is larger than the area of the circle.

Solution. Since the result does not depend on the length units, assume the length of the initial wire is 1. Let U be the length of the first piece. It is uniformly distributed in $(0,1)$. Then

$$P = \mathbf{P}\left\{\left(\frac{U}{4}\right)^2 \geq \pi\left(\frac{1-U}{2\pi}\right)^2\right\}$$
$$= \mathbf{P}\{(4-\pi)U^2 - 8U + 4 < 0\} = \mathbf{P}\{x_1 < U < x_2\},$$

where

$$x_1 = \frac{4-2\sqrt{\pi}}{4-\pi} \approx 0.53, \quad x_2 = \frac{4+2\sqrt{\pi}}{4-\pi} \approx 8.79.$$

So $P = \mathbf{P}\{U > x_1\} = 1 - x_1 = \frac{\sqrt{\pi}}{2+\sqrt{\pi}} \approx 0.47$.

4. Let ξ_1, ξ_2 be two independent Exponentially distributed r.v.'s, their c.d.f.'s are $F_{\xi_1}(x) = 1 - e^{-\lambda_1 x}$ and $F_{\xi_1}(x) = 1 - e^{-\lambda_2 x}$, respectively, for some $\lambda_1, \lambda_2 > 0$. Find:

   (a) the distribution of $\eta_1 = \min\{\xi_1, \xi_2\}$;

   (b) the distribution of $\eta_2 = \max\{\xi_1, \xi_2\}$;

   (c) the joint distribution of η_2 and η_1 in the case when $\lambda_1 = \lambda_2 = \lambda$;

   (d) the conditional distribution of η_2 given η_1 in the case when $\lambda_1 = \lambda_2 = \lambda$.

Solution.

(a) $\mathbf{P}\{\eta_1 > x\} = \mathbf{P}\{\xi_1 > x; \xi_2 > x\} = e^{-\lambda_1}e^{-\lambda_2 x} = e^{-(\lambda_1+\lambda_2)x}$, so that $\eta_1 \sim \text{Exp}(\lambda_1 + \lambda_2)$.

(b) $\mathbf{P}\{\eta_2 \leq x\} = \mathbf{P}\{\xi_1 \leq x; \xi_2 \leq x\} = (1 - e^{-\lambda_1})(1 - e^{-\lambda_2 x})$

(c) If $x > y$ then $\mathbf{P}\{\eta_1 \leq x;\ \eta_2 \leq y\} = \mathbf{P}\{\eta_2 \leq y\}$ which is $F_{\eta_2}(y)$ above. Consider now the case $x \leq y$. Since ξ_1 and ξ_2 are independent, the pair (ξ_1, ξ_2) has joint cdf $F_{(\xi_1,\xi_2)}(x, y) = (1-e^{-\lambda x})(1-\varepsilon^{-\lambda y})$ for $x, y \geq 0$ and 0 otherwise. The set $\{(z_1, z_2) \in \mathbb{R}^2_+ : \min\{z_1, z_2\} \leq x;\ \max\{z_1, z_2\} \leq y\}$ is the union of two rectangles, each having a vertex at the origin and the opposite vertex in the point (x, y) and (y, x), respectively. By the symmetry, the measure of each of them is $F_{(\xi_1,\xi_2)}(x, y)$, the measure of their intersection is $F_{(\xi_1,\xi_2)}(x, x)$. Hence

$$\mathbf{P}\{\eta_1 \leq x;\ \eta_2 \leq y\} = 2F_{(\xi_1,\xi_2)}(x, y) - F_{(\xi_1,\xi_2)}(x, x) = 1 - e^{-2\lambda x} - 2e^{-\lambda y} + 2e^{-\lambda(x+y)}.$$

(d) Obviously, $\mathbf{P}\{\eta_2 > y \mid \eta_1 = x\} = 1$ for all $y < x$. Consider $y \geq x$ and put $z = y - x > 0$. Since ξ_1 and ξ_2 are equally distributed,

$$\begin{aligned}\mathbf{P}\{\eta_2 > y \mid \eta_1 = x\} &= \mathbf{P}\{\xi_2 > y \mid \xi_1 = x; \xi_2 > x\}\,\mathbf{P}\{\xi_1 < \xi_2\} \\ &\quad + \mathbf{P}\{\xi_1 > y \mid \xi_2 = x; \xi_1 > x\}\,\mathbf{P}\{\xi_1 > \xi_2\} \\ &= e^{-\lambda y}/e^{-\lambda x} \cdot 0.5 + e^{-\lambda y}/e^{-\lambda x} \cdot 0.5 = e^{-\lambda z}\end{aligned}$$

which corresponds to the distribution of a r.v. $\eta_1 + \xi$, where $\xi \sim \text{Exp}(\lambda)$.

5. Let $\mu_n = \min\{\xi_1, \dots, \xi_n\}$, where $\xi_1, \dots, \xi_n$ are independent identically distributed non-negative random variables with continuous in the right neighbourhood of 0 density $f(x)$ such that $\lim_{x\downarrow 0} f(x) = f_0$, $0 < f_0 < \infty$ (for example, uniform in $[0, c]$ distributed). Show that $n\mu_n$ converges weakly to the Exponentially distributed random variable with parameter f_0 (i.e. with the cdf $F(x) = 1 - e^{-f_0 x},\ x \geq 0$).

Solution. For any $x \geq 0$,

$$\mathbf{P}\{n\mu_n > x\} = \mathbf{P}\{\min\{\xi_1, \dots, \xi_n\} > x/n\} = \mathbf{P}\left(\cap_{k=1}^n \{\xi_k > x/n\}\right)$$
$$= \Big(\int_{x/n}^{+\infty} f(s)\,ds\Big)^n = \Big(1 - \int_0^{x/n} f(s)\,ds\Big)^n.$$

Since $\int_0^{x/n} f(s)\,ds = f_0\,x/n + o(1/n)$, the last expression is

$$\exp\{n\log(1 - f_0\,x/n + o(1/n))\} \to \exp\{-f_0 x\}.$$

The latter is $1 - F(x)$ for the exponential $\mathrm{Exp}(f_0)$-distribution which is a continuous function. So the limit for all x means the weak convergence.