Foundations of Probability Theory (MVE140 – MSA150)

Wednesday 11th of January 2017 examination questions

You are allowed to use a dictionary (to and from English) and up to a maximum of 2 double-sided pages of your own written notes. This examination has five problems with a maximum of 20 credit points for a fully satisfactory solution, so the maximal total is 100 credit points. To pass the course, you need to score at least 40 points. A member of staff is available at the examination site around 10:30am and 12pm. Tel. 031 772 3574

Examination Questions

1. Let $\xi : \Omega \mapsto \mathbb{R}$ be an arbitrary mapping. Denote by $\sigma(\xi)$ the following collection of subsets of Ω: $\{\xi^{-1}(B) : B \in \mathcal{B}\}$, where B runs through all Borel subsets $\mathcal{B}$ of $\mathbb{R}$.

   (a) Show that $\sigma(\xi)$ is a σ-field[1].

   (b) Find $\sigma(\xi)$ for a mapping ξ taking just two distinct values.

   (c) When does $\sigma(\xi)$ for such ξ coincide with the set 2^Ω of all subsets of Ω?

   Solution. For any $B_1, B_2, \dots \in \mathcal{B}$, $\xi^{-1}(B_1) \cap \xi^{-1}(B_2) = \{\omega : \xi(\omega) \in B_1 \text{ and } \xi(\omega) \in B_2 \text{ and} \dots\} = \xi^{-1}(\cap_i B_i) \in \sigma(\xi)$ since $\cap_i B_i$ is also in $\mathcal{B}$. Also, for any $B \in \mathcal{B}$, $(\xi^{-1}(B))^{\mathbf{c}} = \{\omega : \xi(\omega) \in B^{\mathbf{c}}\} = \xi^{-1}(B^{\mathbf{c}}) \in \sigma(\xi)$. When $\xi(\omega) = c_1$ on $S \subset \Omega$ and hence $\xi(\omega) = c_2$ on $\Omega \setminus S$, then $\xi^{-1}(B) = S$, $\Omega \setminus S = S^{\mathbf{c}}$, Ω or $\emptyset$, depending on whether $c_1 \in B$, $c_2 \in B$, both or none. So $\sigma(\xi) = \{\emptyset, S, S^{\mathbf{c}}, \Omega\}$. This is 2^Ω if Ω itself consists of 2 elements, so that S and $S^{\mathbf{c}}$ are one-point sets.

2. Let $\{\xi_n\}$ be a sequence of positive independent identically distributed (i.i.d.) random variables with finite $a = \mathbf{E}\,\xi$ and $b = \mathbf{E}\,\xi^{-1}$. Let

[1] It is called the σ-field *generated by* ξ.

$S_n = \sum_{i=1}^n \xi_i$. Show that

$$\mathbf{E}(S_m/S_n) = \begin{cases} m/n & \text{if } m \le n; \text{ and} \\ 1 + (m-n)a\,\mathbf{E}(1/S_n) & \text{if } m > n. \end{cases}$$

Solution. For $m \le n$ we have

$$\mathbf{E}\,\frac{S_m}{S_n} = \sum_{i=1}^{n} \mathbf{E}\,\frac{\xi_i}{S_n} = m\,\mathbf{E}\,\frac{\xi_1}{S_n}.$$

But for $m = n$, $1 = n\,\mathbf{E}(\xi_1/S_n)$, hence $\mathbf{E}(S_m/S_n) = m/n$.
For $m > n$,

$$\mathbf{E}\,\frac{S_m}{S_n} = 1 + \sum_{i=n+1}^{m} \mathbf{E}\,\frac{\xi_i}{S_n} = 1 + \sum_{i=n+1}^{m} \mathbf{E}\,\xi_i\,\mathbf{E}\,\frac{1}{S_n} = 1 + (m-n)a\,\mathbf{E}\,S_n^{-1}.$$

3. Let ξ_1, ξ_2 be two independent Exponentially distributed r.v.'s, their c.d.f.'s are $F_{\xi_1}(x) = 1 - e^{-\lambda_1 x}$ and $F_{\xi_1}(x) = 1 - e^{-\lambda_2 x}$, respectively, for some $\lambda_1, \lambda_2 > 0$. Find:

   (a) the distribution of $\eta_1 = \min\{\xi_1, \xi_2\}$;

   (b) the distribution of $\eta_2 = \max\{\xi_1, \xi_2\}$;

   (c) the joint distribution of η_2 and η_1 in the case when $\lambda_1 = \lambda_2 = \lambda$;

   (d) the conditional distribution of η_2 given η_1 in the case when $\lambda_1 = \lambda_2 = \lambda$.

Solution.

(a) $\mathbf{P}\{\eta_1 > x\} = \mathbf{P}\{\xi_1 > x; \xi_2 > x\} = e^{-\lambda_1}e^{-\lambda_2 x} = e^{-(\lambda_1+\lambda_2)x}$, so that $\eta_1 \sim \text{Exp}(\lambda_1 + \lambda_2)$.

(b) $\mathbf{P}\{\eta_2 \le x\} = \mathbf{P}\{\xi_1 \le x; \xi_2 \le x\} = (1 - e^{-\lambda_1})(1 - e^{-\lambda_2 x})$

(c) If $x > y$ then $\mathbf{P}\{\eta_1 \le x;\ \eta_2 \le y\} = \mathbf{P}\{\eta_2 \le y\}$ which is $F_{\eta_2}(y)$ above. Consider now the case $x \le y$. Since ξ_1 and ξ_2 are independent, the pair (ξ_1, ξ_2) has joint cdf $F_{(\xi_1,\xi_2)}(x, y) = (1-e^{-\lambda x})(1-\varepsilon^{-\lambda y})$ for $x, y \ge 0$ and 0 otherwise. The set $\{(z_1, z_2) \in \mathbb{R}^2_+ : \min\{z_1, z_2\} \le x;\ \max\{z_1, z_2\} \le y\}$ is the union of two rectangles, each having a vertex at the origin and the opposite vertex in the point (x, y) and (y, x), respectively. By the symmetry, the measure of each of them is $F_{(\xi_1,\xi_2)}(x, y)$, the measure of their intersection is $F_{(\xi_1,\xi_2)}(x, x)$. Hence

$$\begin{aligned}\mathbf{P}\{\eta_1 \le x;\ \eta_2 \le y\} &= 2F_{(\xi_1,\xi_2)}(x, y) - F_{(\xi_1,\xi_2)}(x, x)\\ &= 1 - e^{-2\lambda x} - 2e^{-\lambda y} + 2e^{-\lambda(x+y)}.\end{aligned}$$

(d) Obviously, $\mathbf{P}\{\eta_2 > y \bigm| \eta_1 = x\} = 1$ for all $y < x$. Consider $y \ge x$ and put $z = y - x > 0$. Since ξ_1 and ξ_2 are equally distributed,

$$\begin{aligned}\mathbf{P}\{\eta_2 > y \bigm| \eta_1 = x\} &= \mathbf{P}\{\xi_2 > y \bigm| \xi_1 = x; \xi_2 > x\}\,\mathbf{P}\{\xi_1 < \xi_2\}\\ &\quad + \mathbf{P}\{\xi_1 > y \bigm| \xi_2 = x; \xi_1 > x\}\,\mathbf{P}\{\xi_1 > \xi_2\}\\ &= e^{-\lambda y}/e^{-\lambda x} \cdot 0.5 + e^{-\lambda y}/e^{-\lambda x} \cdot 0.5 = e^{-\lambda z}\end{aligned}$$

which corresponds to the distribution of a r.v. $\eta_1 + \xi$, where $\xi \sim \text{Exp}(\lambda)$.

4. Suppose that the amounts R_n you win in n-th game of chance are independent identically distributed random variables with a finite mean m and variance σ^2. It is reasonable to assume that $m < 0$. Show that $\mathbf{P}\{(R_1 + \cdots + R_n)/n < m/2\} \to 1$ as $n \to \infty$. What is the moral of this result?

Solution. Let $S_n = R_1 + \dots R_n$. Since $m < 0$ and $\mathbf{E}\, S_n/n = m$, we have that

$$\mathbf{P}\{S_n/n > m/2\} = \mathbf{P}\{S_n/n - m > -m/2\} = \mathbf{P}\{S_n/n - m > |m|/2\}\,.$$

Now by the Chebyshov inequality, the later is at most

$$\mathbf{P}\{|S_n/n - m| > |m|/2\} \le \mathbf{var}(S_n/n)/(m^2/4) = 4\sigma^2/(nm^2) \to 0 \text{ as } n \to \infty.$$

Morale: the probability that your total win after n rounds is positive is vanishing when $m < 0$!

5. Let $\mu_n = \min\{\xi_1, \dots, \xi_n\}$, where $\xi_1, \dots, \xi_n$ are independent uniformly distributed on $[0,1]$ random variables. For which real γ the sequence $n^\gamma \mu_n$ has a weak limit? What are these limiting distributions?
Solution. For any $y \in [0,1)$, $\mathbf{P}\{\mu_n > x\} = \mathbf{P}(\cup_{i=1}^n \{\xi_i > x\}) = (1-x)^n$. Therefore, $\mathbf{P}\{n^\gamma \mu_n > x\} = (1 - n^{-\gamma}x)^n = \exp\{n \log(1 - n^{-\gamma}x\}$. We see that when $\gamma = 1$ the last expression tends to e^{-x} which is $1 - F(x)$ for exponential distribution Exp(1) which is a continuous function. So the limit for all x means the weak convergence. When $\gamma > 1$ the limit is 1 which is the ch.f. of a constant 0. In this case the weak limit is trivial: 0. Finally, there is no limit for $\gamma < 1$.