Foundations of Probability Theory (MVE140 – MSA150)

Tuesday 18th of April 2017 examination questions

You are allowed to use a dictionary (to and from English) and up to a maximum of 2 double-sided pages of your own written notes. This examination has five problems with a maximum of 20 credit points for a fully satisfactory solution, so the maximal total is 100 credit points. To pass the course, you need to score at least 40 points. The examiner, Prof. Sergei Zuyev, is available at the examination site around 10:30am and 12pm. Telephone: 031 772 3020.

Examination Questions

1. Two darts players throw alternately at a board and the first to score a bull's eye (the smallest central circle) wins. On each of their throws player A has probability p_A and player B p_B of success; the results of different throws are independent. If A starts, calculate the probability that he/she wins.
   Solution. Denoting event W_i that A wins at i-th throw ($i = 1, 3, 5, \dots$) and V_j that B wins at j-th throw ($i = 2, 4, 6, \dots$), we have that

$$\begin{aligned}\mathbf{P}\{A \text{ wins}\} &= \mathbf{P}(W_1 \cup \overline{W_1 V_2} W_3 \cup ...) = \mathbf{P}(W_1) + \mathbf{P}(\overline{W_1})\,\mathbf{P}(\overline{V_2})\,\mathbf{P}(W_3) + \dots \\ &= \frac{p_A}{1-(1-p_A)(1-p_B)} = \frac{p_A}{p_A + p_B - p_A p_B}\end{aligned}$$

   Equivalmently, conditionning on the first and second throw, we immediately get

$$\mathbf{P}\{A \text{ wins}\} = p_A + (1-p_A)(1-p_B)\,\mathbf{P}\{A \text{ wins}\}.$$

2. Suppose that X and Y are independent random variables. Show that

$$\mathbf{var}(XY) = \mathbf{var}\,X\ \mathbf{var}\,Y + (\mathbf{E}\,X)^2\,\mathbf{var}\,Y + (\mathbf{E}\,Y)^2\,\mathbf{var}\,X.$$

   Solution. Use independence to write

$$\begin{aligned}\mathbf{var}\,XY &= \mathbf{E}(XY)^2(\mathbf{E}\,XY)^2 = \mathbf{E}\,X^2\,\mathbf{E}\,Y^2(\mathbf{E}\,X)^2(\mathbf{E}\,Y)^2 \\ &= (\mathbf{var}\,X + (\mathbf{E}\,X)^2)(\mathbf{var}\,Y + (\mathbf{E}\,Y)^2)(\mathbf{E}\,X)^2(\mathbf{E}\,Y)^2 \\ &= \mathbf{var}\,X\,\mathbf{var}\,Y + (\mathbf{E}\,X)^2\,\mathbf{var}\,Y + (\mathbf{E}\,Y)^2\,\mathbf{var}\,X.\end{aligned}$$

3. Two symmetric dice are rolled. Find the expected sum of the shown numbers under the condition that two *different* numbers are observed.
Solution. Conditional distribution of the sum ξ:

ξ	3	4	5	6	7	8	9	10	11
$\mathbf{P}$	2/30	2/30	4/30	4/30	6/30	4/30	4/30	2/30	2/30

Hence, $\mathbf{E}[\xi \mid \text{different numbers}] = 7$.

4. Let $\boldsymbol{\xi} = (\xi_1, \xi_2)$ is a random vector in $\mathbb{R}^2$, where ξ_1, ξ_2 are independent standard normal random variables. Let $\rho = \sqrt{\xi_1^2 + \xi_2^2}$ be the radius-vector of $\boldsymbol{\xi}$ and θ is its polar angle.

   (a) Find the joint distribution of the pair (ρ, θ);

   (b) find the marginal distributions of ρ and of θ;

   (c) are ρ and θ independent?

   Solution. Because of independence, the joint density of (ξ_1, ξ_2) is

$$f_{\boldsymbol{\xi}}(x_1, x_2) = \frac{1}{2\pi} e^{-(x_1^2+x_2^2)/2}.$$

   The set $S(r,s) = \{(x_1, x_2) :\ \rho(x_1, x_2) < r,\ \theta(x_1, x_2) < s\}$ is a cone (sector) in $\mathbb{R}^2$, thus after change to polar coordinates $x_1 = u\cos v,\ x_2 = u \sin v$ with the Jacobian u, we have the c.d.f.

$$F_{(\rho,\theta)}(r,s) = \frac{1}{2\pi} \iint_{S(r,s)} e^{(x_1^2+x_2^2)/2}\, dx_1 dx_2 = \frac{1}{2\pi} \int_0^s ds \int_0^r u e^{-u^2/2} du$$
$$= -\frac{s}{2\pi} e^{-u^2/2}|_0^r = \frac{s}{2\pi}(1 - e^{-r^2/2}).$$

   The p.d.f. is then

$$f_{(\rho,\theta)}(r,s) = \frac{\partial^2}{\partial r \partial s} F_{(\rho,\theta)}(r,s) = \frac{1}{2\pi} r e^{-r^2/2},$$

   and we see that it is the product of the constant function $1/(2\pi)$ of $s \in [0, 2\pi)$ and of $re^{-r^2/2}$, $r \geq 0$. Thus the first is the marginal density $f_\theta(s)$ of the polar angle which is uniform on $[0, 2\pi)$ and the marginal density $f_\rho(r) = re^{-r^2/2}$ of $\rho \geq 0$. Thus ρ and θ are independent.

5. Let ξ_n be a random variable taking values $-n$ and n with equal probabilities. Do the following sequences have a weak limit?

   a) $\xi_n/\sqrt{n}$?

   b) $\xi_n/\sqrt{\mathbf{var}\,\xi_n}$?

Solution. The characteristic function: $\varphi_{\xi_n}(t) = 0.5e^{-itn} + 0.5e^{itn} = \cos(tn)$. a) $\varphi_{\xi_n/\sqrt{n}}(t) = \cos(t\sqrt{n})$ has no limit as $n \to \infty$ for any $t \neq 0$, so no weak limit in this case. b) $\mathbf{var}\, \xi_n = n^2$, $\varphi_{\xi_n/n}(t) = \cos(t)$, a continuous function in 0, so the weak limit exists and equals the distribution of a r.v. taking values ± 1 with equal probabilities (actually, all ξ_n/n are distributed like this, hence is the limit).