Foundations of Probability Theory (MVE140 – MSA150)

Friday 13th of January 2016 examination questions

You are allowed to use a dictionary (to and from English) and up to a maximum of 2 double-sided pages of your own written notes. This examination has five problems with a maximum of 20 credit points for a fully satisfactory solution, so the maximal total is 100 credit points. To pass the course, you need to score at least 40 points. The examiner, Prof. Sergei Zuyev, is available at the examination site around 10:30am and 12pm. Telephone: 031 772 3020.

Examination Questions

1. The event A is said to be *repelled* by B if $\mathbf{P}(A \mid B) < \mathbf{P}(A)$, and to be *attracted* by B if $\mathbf{P}(A \mid B) > \mathbf{P}(A)$. Show that

   (a) If B attracts A, then A attracts B;

   (b) B attracts A, then $B^{\mathbf{c}}$ repels A;

   (c) If A attracts B and B attracts C, does A attracts C?

Solution. (a) If B attracts A, then $\mathbf{P}(A \mid B) > \mathbf{P}(A)$, i.e. $\mathbf{P}(AB) > \mathbf{P}(A)\mathbf{P}(B)$. Hence $\mathbf{P}(B \mid A) > \mathbf{P}(B)$ so that A attracts B. (b) By the Full Probability formula,

$$\mathbf{P}(A) = \mathbf{P}(A \mid B)\mathbf{P}(B) + \mathbf{P}(A \mid B^{\mathbf{c}})\mathbf{P}(B^{\mathbf{c}}) > \mathbf{P}(A)\mathbf{P}(B) + \mathbf{P}(A \mid B^{\mathbf{c}})(1 - \mathbf{P}(B))$$

implying

$$\mathbf{P}(A)(1 - \mathbf{P}(B)) > \mathbf{P}(A \mid B^{\mathbf{c}})(1 - \mathbf{P}(B))$$

and hence $\mathbf{P}(A \mid B^{\mathbf{c}}) < \mathbf{P}(A)$. (c) No. It is easier to reformulate the question in a symmetrical form, as in the proof of (a): does $\mathbf{P}(AB) > \mathbf{P}(A)\mathbf{P}(B)$ and $\mathbf{P}(BC) > \mathbf{P}(B)\mathbf{P}(C)$ imply $\mathbf{P}(AC) > \mathbf{P}(A)\mathbf{P}(C)$? Certainly not, if A and C are disjoint. For instance, toss a symmetric coin. If the first toss is Head, toss it again. Let A be event that the second toss is Head, B – the first toss is Head, C – the second toss is Tail. Since A or C can only happen when B happens, $\mathbf{P}(A) = \mathbf{P}(A \mid B)\mathbf{P}(B) = 1/4$. Hence $\mathbf{P}(AB) = 1/4 > \mathbf{P}(A)\mathbf{P}(B) = 1/8$. Similarly for $\mathbf{P}(BC)$. But $AC = \emptyset$ so that $\mathbf{P}(AC) = 0 \not> 1/16 = \mathbf{P}(A)\mathbf{P}(C)$.

2. Random variables ξ and η are independent and equally Bernoulli-distributed. Let χ be the indicator that their sum is an odd number. Is it possible to have ξ and χ independent?
Solution. Let $\xi, \eta \sim \mathrm{Bern}(p)$ for some $p \in (0, 1)$. The sum $\xi + \eta$ then has the Binomial $\mathrm{Bin}(2, p)$ distribution and thus $\chi \sim \mathrm{Bern}(2p(1-p))$. If $\xi = 0$ and $\xi + \eta$ is odd, then $\eta = 1$ so that $\xi = 0, \chi = 0$ happens with probability $(1-p)^2$. Reasoning similarly, $\mathbf{P}\{\xi = 1, \chi = 0\} = p^2$ and $\mathbf{P}\{\xi = 0, \chi = 1\} = \mathbf{P}\{\xi = 1, \chi = 1\} = p(1-p)$. For independence, we must have, in particular, that $\mathbf{P}\{\xi = 1, \chi = 1\} = \mathbf{P}\{\xi = 1\} \times \mathbf{P}\{\chi = 1\}$, i.e. $p(1-p) = p \times 2p(1-p)$ which has a unique solution $p = 1/2$. It is straightforward to verify that the remaining 3 combinations are also the products, so that ξ and χ are independent if and only if $p = 1/2$.

3. Let $\xi_1, \xi_2, \ldots$ be a independent identically distributed (i.i.d.) random variables with characteristic function (ch.f.) $\varphi_\xi(t)$ and let π be a Poisson $\mathrm{Po}(\lambda)$ distributed random variable independent from $\{\xi_n\}$. The sum $\eta = \sum_{n=1}^{\pi} \xi_n$ with random number of terms has the so-called *com-*

pound Poisson distribution (if $\pi = 0$, then $\eta = 0$ by definition). Obviously, when all ξ_n are constant 1, the compound Poisson distribution turns into an ordinary Poisson distribution. Express the ch.f. $\varphi_\eta(t)$ of the sum in terms of $\varphi_\xi(t)$ and find the expectation and the variance of η through $\mathbf{E}\,\xi$ and $\mathbf{var}\,\xi$.

Solution. By the Full-expectation formula and independence of ξ_n's,

$$\begin{aligned}
\varphi_\eta(t) &= \sum_{n=0}^{\infty} \big[\,\mathbf{E}[\exp\{it\sum_{i=1}^{n}\xi_i\} \;\big|\; \pi = n\}]\big]\mathbf{P}\{\pi = n\} \\
&= \sum_{n=0}^{\infty} \big[\prod_{i=1}^{n}\mathbf{E}[\exp\{it\xi_i\} \;\big|\; \pi = n\}]\frac{\lambda^n}{n!}e^{-\lambda} \\
&= \sum_{n=0}^{\infty} \varphi_\xi^n(t)\frac{\lambda^n}{n!}e^{-\lambda} = e^{\varphi_\xi^n(t)\lambda}e^{-\lambda} = \exp\{\lambda(\varphi_\xi(t) - 1)\}.
\end{aligned}$$

Thus, $\varphi_\eta'(t) = \lambda\varphi_\xi'(t)\varphi_\eta(t)$ and $\varphi_\eta''(t) = \lambda\varphi_\xi''(t)\varphi_\eta(t) + \lambda^2(\varphi_\xi'(t))^2\varphi_\eta(t)$. Since $\varphi_\eta(0) = 1$, $\varphi_\eta'(0) = i\,\mathbf{E}\,\eta$ and $\varphi_\eta''(0) = -\,\mathbf{E}\,\eta^2$, we have that $\mathbf{E}\,\eta = \lambda\,\mathbf{E}\,\xi$ and $\mathbf{E}\,\eta^2 = \lambda\,\mathbf{E}\,\xi^2 + \lambda^2(\mathbf{E}\,\xi)^2$ so that $\mathbf{var}\,\eta = \lambda\,\mathbf{E}\,\xi^2 = \lambda(\mathbf{var}\,\xi + (\mathbf{E}\,\xi)^2)$.

4. Two points ξ and η are independently and uniformly chosen on a segment $[0, a]$. Find

   (a) the distribution (in terms of c.d.f. or p.d.f.) of the distance $\delta = |\xi - \eta|$ between them;

   (b) the mean distance $\mathbf{E}\,\delta$;

   (c) the conditional distribution of δ given $\xi = x$;

   (d) the conditional expectation $\mathbf{E}[\delta \;\big|\; \xi]$;

   (e) its expectation $\mathbf{E}\,\mathbf{E}[\delta \;\big|\; \xi]$. Does this coinside with $\mathbf{E}\,\delta$?

Solution. (a) The vector (ξ, η) is uniformly distributed in the square $Q = [0, a]$. The event $\delta \le t$ is the same as the event that the point (ξ, η) lies in the strip S_t between the lines $y = x - t$ and $y = x + t$. Then $F_\delta(t) = \mathbf{P}\{\delta \le t\} = |S_t \cap Q|/|Q| = (t^2 + 2t(a-t))/a^2 = (2at - t^2)/a^2$ if $0 \le t < a$ and 1 if $t \ge a$. (b) $\mathbf{E}\,\delta = \int_0^a t F'_\delta(t) dt = a/3$, as it can be checked. (c) When $\xi = x$, $|\xi - \eta| \le t$ means that η is in t-neighbourhood of x, so that $\mathbf{P}\{\delta \le t \mid \xi = x\} = |[x-t, x+t] \cap [0, a]|/|[0, a]|$ which is $2t/a$ for $t \le x$ and $(t+x)/a$ for $x \le t \le a - x$ and 1 for larger t. Hence the density $f_{\delta|\xi=x}(t|x) = 2/a\, \mathbb{1}_{(0,x)}(t) + 1/a\, \mathbb{1}_{(x,a-x)}(t)$. (d) Then $\mathbf{E}[\delta \mid \xi = x] = \int_0^{a-x} t f_{\delta|\xi=x}(t|x)\, dt = x^2/a - x + a/2$ so that $\mathbf{E}[\delta \mid \xi] = \xi^2/a - \xi + a/2$ (e) with expectation $a/3$. This is the same as $\mathbf{E}\,\delta$ above.

5. Let $\mu_n = \min\{\xi_1, \ldots, \xi_n\}$, where $\xi_1, \ldots, \xi_n$ are independent continuously distributed random variables with density $f_\xi(x)$ such that $f_\xi(x) = 0$ for all $x < 0$ and $\lim_{x \downarrow 0} f_\xi(x) = f_\xi(+0) > 0$. Show that $n\mu_n$ converges weakly to the Exponentially distributed random variable with parameter $f_\xi(+0)$.
Solution. Let $F_\xi(x)$ be the c.d.f. of ξ. For any $y \in [0, 1)$, $\mathbf{P}\{\mu_n > y\} = \mathbf{P}\{\min\{\xi_1, \ldots, \xi_n\} > y\} = \mathbf{P}\big(\cap_{k=1}^n \{\xi_k > y\}\big) = (1 - F_\xi(y))^n$. Since ξ is continuous, we have that $F_\xi(y) = F_\xi(0) + yF'_\xi(+0) + o(y) = yf_\xi(+0) + o(y)$. Therefore, for any $x \ge 0$, $\mathbf{P}\{n\mu_n > x\} = (1 - xf_\xi(+0)/n + o(1/n))^n \to e^{-xf_\xi(+0)}$. The latter is $1 - F(x)$ for the exponential $\mathrm{Exp}(f_\xi(+0))$ distribution which is a continuous function. So the limit for all x means the weak convergence.