Foundations of Probability Theory (MVE140 – MSA150)

Thursday 7th of April 2016 examination questions

You are allowed to use a dictionary (to and from English) and up to a maximum of 2 double-sided pages of your own written notes. This examination has five problems with a maximum of 20 credit points for a fully satisfactory solution, so the maximal total is 100 credit points. To pass the course, you need to score at least 40 points. The examiner, Prof. Sergei Zuyev, is available at the examination site around 3:30pm and 5pm. Telephone: 031 772 3020.

Examination Questions

1. A regular tetrahedron (pyramid) is tossed. One of its faces is coloured red, the other blue, the third green and the forth has all three of these colours. Let R, G, B denote the events that the face touching the ground contains red, green or blue colour respectively. Are these events:

   (a) pariwise independent?

   (b) mutually independent?

   Solution. By symmetry, $\mathbf{P}(R) = \mathbf{P}(G) = \mathbf{P}(B) = 1/4 + 1/4 = 1/2$, $\mathbf{P}RG = \mathbf{P}(RGB) = 1/4 = 1/2 \times 1/2 = \mathbf{P}(R)\mathbf{P}(G)$, i.e. R and G are pairwise independent. Similarly for the other pairs. However, $\mathbf{P}(RGB) = 1/4 \neq \mathbf{P}(R)\mathbf{P}(G)\mathbf{P}(B) = 1/8$, so the events are not mutually independent.

2. An organiser of a TV show suggests a participant to choose between two urns and draw one ball from it. If the participant draws a white ball he or she wins a price, but if the ball is black, the participant quits the show. There are altogether 4 balls: 2 white and 2 black. How the organiser should in advance distribute the balls over the two urns (not all in the same urn) so that to minimise the chance of a participant to win?

Solution. The numbers of balls in the urns can be either 2+2 or 1+3. The case 2+2 is totally symmetric with respect to the colour change so the probability to get a white ball (as well as a black one) is 1/2. Consider a configuration 1+3 with one white ball in one urn. Let W be event that a white ball is chosen and A (A_3) be the event that the urn with 1 ball (resp., 3 balls) is chosen to draw from. Using the Full probability formula,

$$\mathbf{P}(W) = \mathbf{P}(W|A_1)\mathbf{P}(A_1) + \mathbf{P}(W|A_3)\mathbf{P}(A_3) = 1 \cdot 1/2 + 1/3 \cdot 1/2 = 2/3.$$

If only one black ball is put in one of the urns, we have

$$\mathbf{P}(W) = 0 \cdot 1/2 + 2/3 \cdot 1/2 = 1/3 < 1/2$$

so this is the strategy to take.

3. Let ξ, η be independent random variables. Prove inequality

$$\mathbf{var}(\xi\eta) \geq \mathbf{var}\,\xi \;\; \mathbf{var}\,\eta.$$

When does this relation turn into equality?

Solution. From the definition of the variance and using independence,

$$\mathbf{var}(\xi\eta) - \mathbf{var}\,\xi \;\; \mathbf{var}\,\eta = (\mathbf{E}\,\xi)^2\,\mathbf{var}\,\eta + (\mathbf{E}\,\eta)^2\,\mathbf{var}\,\xi \geq 0$$

Thus the equality is either when any of the variables is a constant 0, or both are constants or both have zero expectation.

4. Let ξ and η be independent random variables each having Exponential $\text{Exp}(\lambda)$ distribution. Find the joint density of the pair $(\xi, \xi + \eta)$ and deduce that the conditional density of ξ given $\xi + \eta = t$ corresponds to the uniform distribution on $(0, t)$. In other words, knowing $\xi + \eta$ bears no information on the value of ξ!

Solution. The density of ξ (and also η) is $f_\xi(x) = \lambda e^{-\lambda x} = f_\eta(x),\ x \geq 0$. The conditional density of $\xi + \eta$ given $\xi = x$ corresponds to the density of $x + \eta$ so it is $f_{\xi+\eta|\xi}(y|x) = \lambda e^{-\lambda(y-x)}$ for $y \geq x$ and 0 otherwise. Thus the joint density is

$$f_{\xi,\xi+\eta}(x,y) = f_{\xi+\eta|\xi}(y|x) f_\xi(x) = \lambda^2 e^{-\lambda y}\, \mathbb{1}\{y \geq x\}.$$

Therefore,

$$\begin{aligned} f_{\xi|\xi+\eta}(x|y) &= f_{\xi,\xi+\eta}(x,y)/f_{\xi+\eta}(y) \\ &= f_{\xi,\xi+\eta}(x,y) \left[\int_0^y f_{\xi,\xi+\eta}(x,y)\, dx \right]^{-1} \\ &= \lambda^2 e^{-\lambda y}\, \mathbb{1}\{y \geq x\} \left[y \lambda^2 e^{-\lambda y}\, \mathbb{1}\{y \geq x\} \right]^{-1} = 1/y. \end{aligned}$$

The density (a function of x!) is a constant $1/y$ on the interval $[0, y]$, i.e. the distribution is uniform.

5. Let $\{\xi_n\}$ be a sequence of random variables with the distribution symmetrical with respect to point a: ξ_n takes values $-n^\alpha - a$ and $n^\alpha - a$ with some α with equal probabilities. Characterise the sequences of normalising constants $\{c_n\}$ for which the sequence $c_n \xi_n$ has a weak limit. When does this limit is non-trivial (i.e. it is not a constant)?
Solution. The characteristic function: $\varphi_{\xi_n}(t) = e^{iat}(0.5e^{-itn^\alpha} + 0.5e^{itn^\alpha}) = e^{iat}\cos(tn^\alpha)$. Thus $\varphi_{c_n\xi_n}(t) = e^{iac_n t}\cos(tc_n n^\alpha)$ which has a limit as $n \to \infty$ iff both terms have a limit, i.e. when $c_n n^\alpha \to C_1 < \infty$ and $c_n \to C_2 < \infty$. Thus, either $\alpha < 0$ and $0 \leq C_2 < \infty$ or $\alpha = 0$ and $C_2 = 0$ or $\alpha > 0$ and $C_2 = 0$, then $\varphi_{c_n\xi_n}(t) \to e^{iaC_2 t}$, i.e. the limit is trivial corresponding to the constant aC_2. Alternatively, either $\alpha = 0$ and $0 < C_2 < \infty$, then $\varphi_{c_n\xi_n}(t) \to e^{iaC_2 t}(0.5e^{-it} + 0.5e^{it})$, i.e. the limit is a random variable taking values $aC_2 - 1$ and $aC_2 + 1$ with equal probabilities. Or $\alpha > 0$ and $0 < C_1 < \infty$, in which case $\varphi_{c_n\xi_n}(t) \to (0.5e^{-itC_1} + 0.5e^{itC_1})$, i.e. the limit is a symmetric random variable taking values $-C_1$ and C_1 with equal probabilities.