Foundations of Probability Theory (MVE140 – MSA150)

Friday 16th of January 2015 examination questions

You are allowed to use a dictionary (to and from English) and up to a maximum of 2 double-sided pages of your own written notes. This examination has five problems with a maximum of 20 credit points for a fully satisfactory solution, so the maximal total is 100 credit points. To pass the course, you need to score at least 40 points. The examiner, Prof. Sergei Zuyev, is available at the examination site around 10:30am and 12pm. Telephone: 031 772 3020.

Examination Questions

1. In a certain population, the probability that the first child born to a woman is a boy is 0.51. It was also noticed that the probability that the second child is of the same sex as the first one is 0.55. The second child of a randomly selected woman is a girl. What is the probability that her first child is a boy?

2. In your pocket there is a random number N of coins, where N is Poisson distributed with parameter λ. You toss all of them, the Head showing with probability p for each coin. Show that the distribution of the number of Heads shown is Poisson $\mathrm{Po}(p\lambda)$.

3. Find the density of the random variable $\eta = e^{\xi}$, where ξ is a standard Normal random variable.

4. Let ξ_1, ξ_2 be two independent Exponentially distributed r.v.'s, their c.d.f.'s are $F_{\xi_1}(x) = 1 - e^{-\lambda_1 x}$ and $F_{\xi_1}(x) = 1 - e^{-\lambda_2 x}$, respectively, for some $\lambda_1, \lambda_2 > 0$. Find:

   (a) the distribution of $\eta_1 = \min\{\xi_1, \xi_2\}$;

   (b) the distribution of $\eta_2 = \max\{\xi_1, \xi_2\}$;

   (c) the joint distribution of η_2 and η_1 in the case when $\lambda_1 = \lambda_2 = \lambda$;

   (d) the conditional distribution of η_2 given η_1 when $\lambda_1 = \lambda_2 = \lambda$.

5. Let ξ_λ be a random variable with Poisson distribution $\mathrm{Po}(\lambda)$. Show that the distribution of the random variable $\eta_\lambda = (\xi_\lambda - \mathbf{E}\,\xi_\lambda)/\sqrt{\mathbf{var}\,\xi_\lambda}$ weakly converges as $\lambda \to \infty$ to the standard Normal distribution $\mathcal{N}(0,1)$.

Lösningar!

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Examination Questions

1. In a certain population, the probability that the first child born to a woman is a boy is 0.51. It was also noticed that the probability that the second child is of the same sex as the first one is 0.55. The second child of a randomly selected woman is a girl. What is the probability that her first child is a boy?
   Solution. Let B_1 be the event that the first child is a boy and B_2 is that the second child is a boy. By the Bayes formula, $\mathbf{P}(B_1 \mid B_2{}^c) = \mathbf{P}(B_2{}^c \mid B_1)\mathbf{P}(B_1)/(\mathbf{P}(B_2{}^c \mid B_1)\mathbf{P}(B_1) + \mathbf{P}(B_2{}^c \mid B_1{}^c)\mathbf{P}(B_1{}^c)) = 0.45 \cdot 0.51/(0.45 \cdot 0.51 + 0.55 \cdot 0.49) = 0.4599$.

2. In your pocket there is a random number N of coins, where N is Poisson distributed with parameter λ. You toss all of them, the Head showing with probability p for each coin. Show that the distribution of the number of Heads shown is Poisson $\mathrm{Po}(p\lambda)$.

Solution. Given $N = n$ the number η of coins showing Head has Binomial $\mathrm{Bin}(n,p)$ distribution. Therefore, by the Full Probability,

$$\mathbf{P}\{\eta = k\} = \sum_{n=k}^{\infty} \mathbf{P}\{\eta = k \mid N = n\}\mathbf{P}\{N = n\} = \sum_{n=k}^{\infty} \binom{n}{k} p^k (1-p)^{n-k} \frac{\lambda^n}{n!} e^{-\lambda}$$

$$= \frac{p^k \lambda^k}{k!} e^{-\lambda} \sum_{n=k}^{\infty} \frac{(1-p)^{n-k}\lambda^{n-k}}{(n-k)!} = \frac{p^k\lambda^k}{k!} e^{-\lambda} e^{(1-p)\lambda} = \frac{(p\lambda)^k}{k!} e^{-p\lambda},$$

which is the Poisson distribution with parameter $p\lambda$.

3. Find the density of the random variable $\eta = e^{\xi}$, where ξ is a standard Normal random variable.
Solution. $f_\xi(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$, hence for $x > 0$,

$$F_\eta(x) = \mathbf{P}\{e^{\xi} \le x\} = \mathbf{P}\{e^{\xi} \le \log x\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\log x} e^{-t^2/2} dt$$

and $F_\eta(x) = 0$ for $x \le 0$. Putting $t = \log s$, we get

$$F_\eta(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \frac{e^{-\log^2 s/2}}{s} ds$$

so that $f_\eta(x) = F'_\eta(x) = x^{-1} e^{-\log^2 x/2}$ for $x > 0$ and 0 for $x < 0$.
NB. This distribution is called *log-normal.*

4. Let ξ_1, ξ_2 be two independent Exponentially distributed r.v.'s, their c.d.f.'s are $F_{\xi_1}(x) = 1 - e^{-\lambda_1 x}$ and $F_{\xi_1}(x) = 1 - e^{-\lambda_2 x}$, respectively, for some $\lambda_1, \lambda_2 > 0$. Find:

   (a) the distribution of $\eta_1 = \min\{\xi_1, \xi_2\}$;

   (b) the distribution of $\eta_2 = \max\{\xi_1, \xi_2\}$;

   (c) the joint distribution of η_2 and η_1 in the case when $\lambda_1 = \lambda_2 = \lambda$;

   (d) the conditional distribution of η_2 given η_1 when $\lambda_1 = \lambda_2 = \lambda$.

Solution.

(a) $\mathbf{P}\{\eta_1 > x\} = \mathbf{P}\{\xi_1 > x; \xi_2 > x\} = e^{-\lambda_1 x} e^{-\lambda_2 x} = e^{-(\lambda_1+\lambda_2)x}$, so that $\eta_1 \sim \operatorname{Exp}(\lambda_1 + \lambda_2)$.

(b) $\mathbf{P}\{\eta_2 \leq x\} = \mathbf{P}\{\xi_1 \leq x; \xi_2 \leq x\} = (1 - e^{-\lambda_1 x})(1 - e^{-\lambda_2 x})$

(c) If $x > y$ then $\mathbf{P}\{\eta_1 \leq x;\ \eta_2 \leq y\} = \mathbf{P}\{\eta_2 \leq y\}$ which is $F_{\eta_2}(y)$ above. Consider now the case $x \leq y$. Since ξ_1 and ξ_2 are independent, the pair (ξ_1, ξ_2) has joint cdf $F_{(\xi_1,\xi_2)}(x, y) = (1-e^{-\lambda x})(1-\varepsilon^{-\lambda y})$ for $x, y \geq 0$ and 0 otherwise. The set $\{(z_1, z_2) \in \mathbb{R}^2_+ : \min\{z_1, z_2\} \leq x;\ \max\{z_1, z_2\} \leq y\}$ is the union of two rectangles, each having a vertex at the origin and the opposite vertex in the point (x, y) and (y, x), respectively. By the symmetry, the measure of each of them is $F_{(\xi_1,\xi_2)}(x, y)$, the measure of their intersection is $F_{(\xi_1,\xi_2)}(x, x)$. Hence

$$\mathbf{P}\{\eta_1 \leq x;\ \eta_2 \leq y\} = 2F_{(\xi_1,\xi_2)}(x, y) - F_{(\xi_1,\xi_2)}(x, x) = 1-e^{-2\lambda x}-2e^{-\lambda y}+2e^{-\lambda(x+y)}.$$

(d) Obviously, $\mathbf{P}\{\eta_2 > y \mid \eta_1 = x\} = 1$ for all $y < x$. Consider $y \geq x$ and put $z = y - x > 0$. Since ξ_1 and ξ_2 are equally distributed,

$$\begin{aligned}
\mathbf{P}\{\eta_2 > y \mid \eta_1 = x\} &= \mathbf{P}\{\xi_2 > y \mid \xi_1 = x; \xi_2 > x\}\mathbf{P}\{\xi_1 < \xi_2\} \\
&\quad + \mathbf{P}\{\xi_1 > y \mid \xi_2 = x; \xi_1 > x\}\mathbf{P}\{\xi_1 > \xi_2\} \\
&= e^{-\lambda y}/e^{-\lambda x} \cdot 0.5 + e^{-\lambda y}/e^{-\lambda x} \cdot 0.5 = e^{-\lambda z}
\end{aligned}$$

which corresponds to the distribution of a r.v. $\eta_1+\xi$, where $\xi \sim \operatorname{Exp}(\lambda)$.

5. Let ξ_λ be a random variable with Poisson distribution $\operatorname{Po}(\lambda)$. Show that the distribution of the random variable $\eta_\lambda = (\xi_\lambda - \mathbf{E}\,\xi_\lambda)/\sqrt{\mathbf{var}\,\xi_\lambda}$ weakly converges as $\lambda \to \infty$ to the standard Normal distribution $\mathcal{N}(0, 1)$.

Solution. We have for the Poisson distribution that $\mathbf{E}\,\xi_\lambda = \mathbf{var}\,\xi_\lambda = \lambda$ and the characteristic function $\varphi_{\xi_\lambda}(t) = \exp\{\lambda(e^{it} - 1)\}$. By the Shift theorem,

$$\begin{aligned}\varphi_{\eta_\lambda}(t) = e^{-it\sqrt{\lambda}}\varphi_{\xi_\lambda}(t/\sqrt{\lambda}) = \exp\{-it\sqrt{\lambda} + \lambda(e^{it/\sqrt{\lambda}} - 1)\}\\ = \exp\{-t^2/2 + o(t^2)\} \to e^{-t^2/2}\end{aligned}$$

which is the ch.f. of $\mathcal{N}(0,1)$ law.