Foundations of Probability Theory (MVE140 – MSA150)

Friday 20th of December 2013 examination questions

You are allowed to use a dictionary (to and from English) and up to a maximum of 2 double-sided A4 pages of your own notes. This examination has five problems with a maximum of 20 credit points for a fully satisfactory solution, so the maximal total is 100 credit points. To pass the course, you need to score at least 40 points. A staff representative, Alexey Lindo, will be available at the examination site around 10:30am. Telephone: 031 772 8294.

Examination Questions

1. In modern mobile communications, the voice is first encoded by the mobile into a sequence of bits: 0's and 1's which is then transmitted to the base station. Assume that the coding algorithm is such that 0's and 1's are sent in the proportion 4:3. A high number of calls originating from the sports arena during a football match cases interference resulting in a very erratic transmission when a bit 1 is received as 0 with probability 1/4, whereas a bit 0 is taken for a 1 by the base station with probability 1/3. If a bit 1 is received, that is the probability that it has actually been a 1 sent?

2. A piece of wire is cut into two pieces at an arbitrary point. One piece is bent into a square, the other piece – into a circle. Find the probability that the area of the square is larger than the area of the circle.

3. Let $\xi_1, \xi_2, \dots$ be a sequence of independent Bernoulli-distributed with parameter p random variables and ν be a Poisson distributed random variable with parameter λ independent of $\xi_n' s$. Find the characteristic function of the sum $\zeta = \sum_{n=1}^{\nu} \xi_n$ (the sum is understood as 0, if $\nu = 0$). Find the expectation and the variance of ζ.

4. The joint distribution of two random variables (ξ, η) has density $f_{(\xi,\eta)}(x, y) = e^{-x-y}$, $x, y \geq 0$ and 0 otherwise.

   a) Find the joint density of the vector $(\xi + \eta, \xi/\eta)$.

b) Prove that $\xi + \eta$ and ξ/η are independent.

5. Gamma distribution $\Gamma(n, \lambda)$ is the distribution of a sum of n independent exponentially $\mathrm{Exp}(\lambda)$ distributed random variables. Find

   (a) its characteristic function;

   (b) its mean and its variance;

   (c) For $a > 0$ find the weak limit of the sequence $\{\zeta_n = \xi_n - \sqrt{n}/a\}$, where ξ_n are random variables distributed as $\Gamma(n, a\sqrt{n})$.

SOLUTIONS

Foundations of Probability Theory (MVE140 – MSA150)

Friday 20th of December 2013 examination questions

You are allowed to use a dictionary (to and from English) and up to a maximum of 2 double-sided A4 pages of your own notes. This examination has five problems with a maximum of 20 credit points for a fully satisfactory solution, so the maximal total is 100 credit points. To pass the course, you need to score at least 40 points. A staff representative, Alexey Lindo, will be available at the examination site around 10:30am. Telephone: 031 772 8294.

Examination Questions

1. In modern mobile communications, the voice is first encoded by the mobile into a sequence of bits: 0's and 1's which is then transmitted to the base station. Assume that the coding algorithm is such that 0's and 1's are sent in the proportion 4:3. A high number of calls originating from the sports arena during a football match cases interference resulting in a very erratic transmission when a bit 1 is received as 0 with probability 1/4, whereas a bit 0 is taken for a 1 by the base station with probability 1/3. If a bit 1 is received, that is the probability that it has actually been a 1 sent?
   Solution. Apply the Bayes theorem to get answer 27/43

2. A piece of wire is cut into two pieces at an arbitrary point. One piece is bent into a square, the other piece – into a circle. Find the probability that the area of the square is larger than the area of the circle.

Solution. Since the result does not depend on the length units, assume the length of the initial wire is 1. Let U be the length of the first piece. It is uniformly distributed in $(0, 1)$. Then

$$P = \mathbf{P}\Big\{\Big(\frac{U}{4}\Big)^2 \geq \pi\Big(\frac{1-U}{2\pi}\Big)^2\Big\}$$
$$= \mathbf{P}\{(4-\pi)U^2 - 8U + 4 < 0\} = \mathbf{P}\{x_1 < U < x_2\},$$

where

$$x_1 = \frac{8 - \sqrt{64 - 16(4-\pi)}}{2(4-\pi)} \approx 0.53, \quad x_2 = \frac{8 + \sqrt{64 - 16(4-\pi)}}{2(4-\pi)} \approx 8.79.$$

So $P = \mathbf{P}\{U > x_1\} = 1 - x_1 \approx 1 - 0.53 = 0.47$.

3. Let $\xi_1, \xi_2, \ldots$ be a sequence of independent Bernoulli-distributed with parameter p random variables and ν be a Poisson distributed random variable with parameter λ independent of ξ_n's. Find the characteristic function of the sum $\zeta = \sum_{n=1}^{\nu} \xi_n$ (the sum is understood as 0, if $\nu = 0$). Find the expectation and the variance of ζ.
Solution. $\varphi_\xi(t) = 1 - p + pe^{it}$,

$$\varphi_\zeta(t) = \sum_{n=0}^{\infty}(\mathbf{E}\, e^{it\xi})^n \lambda^n/(n!)\, e^{-\lambda} = \sum_{n=0}^{\infty}(\varphi_\xi(t)\lambda)^n/(n!)\, e^{-\lambda} = e^{\lambda(\varphi_\xi(t)-1)}$$
$$= e^{\lambda p(e^{it}-1)} = 1 + it\lambda p - \frac{t^2}{2}(\lambda p + \lambda^2 p^2) + o(t^2).$$

Hence $\mathbf{E}\,\zeta = \lambda p$, $\mathbf{E}\,\zeta^2 = \lambda p + \lambda^2 p^2$ and $\mathbf{var}\,\zeta = \lambda p$ also.

4. The joint distribution of two random variables (ξ, η) has density $f_{(\xi,\eta)}(x, y) = e^{-x-y}$, $x, y \geq 0$ and 0 otherwise.

   a) Find the joint density of the vector $(\xi + \eta, \xi/\eta)$.

   b) Prove that $\xi + \eta$ and ξ/η are independent.

Solution. The joint c.d.f. of the pair $(\xi+\eta, \xi/\eta)$ is

$$F_{(\xi+\eta,\xi/\eta)}(u,v) = \mathbf{P}\{\xi+\eta \le u, \xi/\eta \le v\} \tag{1}$$

thus, representing the pair as a vector in the first quadrant of the plane, (ξ,η) satisfying (1) should lie in the triangle bounded by the lines $\{x=0\}$, $\{y=u-x\}$, $\{y=x/v\}$. The vertices of the triangle are $(0,0)$, $(0,u)$, $(uv(1+v)^{-1}, u(1+v)^{-1}) = (x_0,y_0)$. Integration of the density $f_{(\xi,\eta)}(x,y)$ over this triangle gives

$$F_{(\xi+\eta,\xi/\eta)}(u,v) = \int_0^{x_0} dx \int_{y=x/v}^{y=u-x} e^{-x-y}dy = \frac{1-e^{-u}-ue^{-u}}{1+1/v}. \tag{2}$$

Thus

$$f_{(\xi+\eta,\xi/\eta)}(u,v) = \frac{\partial^2}{\partial u \partial v} F_{(\xi+\eta,\xi/\eta)}(u,v) = \frac{ue^{-u}}{(1+v)^2}. \tag{3}$$

Since in (1)

$$F_{(\xi+\eta,\xi/\eta)}(u,v) = F_{(\xi+\eta,\xi/\eta)}(u,+\infty)F_{(\xi+\eta,\xi/\eta)}(+\infty,v) = F_{\xi+\eta}(u)F_{\xi/\eta}(v),$$

random variables $\xi+\eta$ and ξ/η are independent.

5. Gamma distribution $\Gamma(n,\lambda)$ is the distribution of a sum of n independent exponentially $\mathrm{Exp}(\lambda)$ distributed random variables. Find
   (a) its characteristic function;
   (b) its mean and its variance;
   (c) For $a>0$ find the weak limit of the sequence $\{\zeta_n = \xi_n - \sqrt{n}/a\}$, where ξ_n are random variables distributed as $\Gamma(n, a\sqrt{n})$.

Solution. For $\xi_1 \sim \mathrm{Exp}(\lambda)$ the ch.f. is $\varphi_1(t) = \lambda/(\lambda - it)$, with mean $1/\lambda$ and variance $1/\lambda^2$ (e.g., by differentiating the ch.f. at $t=0$: $\mathbf{E}\,\xi^k = i^k\varphi_1^{(k)}(0)$). Hence for Gamma-distributed ξ, $\varphi_\xi(t) = \lambda^n/(\lambda-it)^n$, $\mathbf{E}\,\xi = -i\varphi_\xi'(0) = n/\lambda$, $\mathbf{var}\,\xi = n/\lambda^2$ as the sum of n independent exponentially distributed r.v.'s. Now, either use CLT for this sum: its mean is $\sqrt{n}/a$ and variance is $1/a^2$, so $a\zeta_n \xrightarrow{w} \mathcal{N}(0,1)$ implying $\zeta_n \xrightarrow{w} \mathcal{N}(0,a^{-2})$. Or show directly that $\varphi_{\zeta_n}(t) = e^{-it\sqrt{n}/a}a^n n^{n/2}/(a\sqrt{n}-it)^n \to e^{-t^2/(2a^2)}$ for all $t \in \mathbb{R}$.