Foundations of Probability Theory (MVE140 – MSA150)

Friday 18th of January 2013 resit examination questions

You are allowed to use a dictionary (to and from English), a university approved calculator and up to a maximum of 3 double-sided pages of your own written notes. This examination has five problems with a maximum of 20 credit points for a fully satisfactory solution, so the maximal total is 100 credit points. To pass the course, you need to score at least 40 points. The examiner, Prof. Sergei Zuyev, will be available at the examination site around 10:30am and 12pm. Telephone: 031 772 3020.

Examination Questions

1. n people at a supermarket, including Mr. X and Ms. Y, rush towards a newly opened cash till and get into the queue in the order of their arrival (i.e. randomly). What is the probability that between Mr. X and Ms. Y will be exactly $k = 0, \ldots, n-2$ people standing in the queue?

2. Let A and B be two events such that $\mathbf{P}(B) > 0$. Show that if $\mathbf{P}(A \mid B) = \mathbf{P}(A \mid B^c)$ then A and B are independent.

3. Two points p and q are chosen randomly and uniformly from $[-1, 1]$. What is probability that the equation $x^2 + px + q = 0$ has only real roots?

4. Let ξ be a non-negative random variable. Show that
$$\sum_{k=1}^{\infty}(k-1)\,\mathbb{I}_{A_k} \leq \xi < \sum_{k=1}^{\infty} k\,\mathbb{I}_{A_k},$$
where $A_k = \{\omega :\ k-1 \leq \xi(\omega) < k\}$ and deduce that
$$\sum_{k=1}^{\infty}\mathbf{P}\{\xi \geq k\} \leq \mathbf{E}\,\xi < 1 + \sum_{k=1}^{\infty}\mathbf{P}\{\xi \geq k\}.$$

5. Let $\mu_n = \max\{\xi_1, \ldots, \xi_n\}$, where $\xi_1, \ldots, \xi_n$ are independent uniformly distributed on $[0,1]$ random variables. Show that $n(1 - \mu_n)$ converges weakly to the Exponentially distributed random variable with parameter 1 (i.e. with the cdf $F(x) = 1 - e^{-x}$, $x \geq 0$.

LÖSNINGAR!

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Examination Questions

1. n people at a supermarket, including Mr. X and Ms. Y, rush towards a newly opened cash till and get into the queue in the order of their arrival (i.e. randomly). What is the probability that between Mr. X and Ms. Y will be exactly $k = 0, \dots, n-2$ people standing in the queue?
   Solution. There are $n!$ ways to place n people into the queue. There are $n-k-1$ positions in the queue separated by k other people and there are 2 ways to place X and Y in these places. Other $n-2$ people can be distributed over the remaining $n-2$ places in $(n-2)!$ ways. Thus the probability is $2(n-k-1)(n-2)!/n! = 2(1-k/(n-1))/n$.

2. Let A and B be two events such that $\mathbf{P}(B) > 0$. Show that if $\mathbf{P}(A \mid B) = \mathbf{P}(A \mid B^c)$ then A and B are independent.
   Solution. We have that
   $$\frac{\mathbf{P}(AB)}{P(B)} = \frac{\mathbf{P}(AB^c)}{P(B^c)} = \frac{\mathbf{P}(A) - \mathbf{P}(AB)}{1 - P(B)}.$$
   so that $\mathbf{P}(AB)(1-\mathbf{P}(B)) = \mathbf{P}(B)(\mathbf{P}(A) - \mathbf{P}(AB))$ implying $\mathbf{P}(AB) = \mathbf{P}(A)\mathbf{P}(B)$, i.e. independence.

3. Two points p and q are chosen randomly and uniformly from $[-1,1]$. What is probability that the equation $x^2 + px + q = 0$ has only real roots?
   Solution. The roots are real if the discriminant of the equation is non-negative, so that $p^2 - 4q \geq 0$. This means that a uniformly distributed point (p,q) in the square $[-1,1]^2$ lies below the parabola $q = p^2/4$. The corresponding area is $13/6$, thus the probability is $13/24$.

4. Let ξ be a non-negative random variable. Show that
   $$\sum_{k=1}^{\infty}(k-1)\,\mathbb{1}_{A_k} \leq \xi < \sum_{k=1}^{\infty} k\,\mathbb{1}_{A_k},$$

where $A_k = \{\omega : \; k-1 \le \xi(\omega) < k\}$ and deduce that

$$\sum_{k=1}^{\infty} \mathbf{P}\{\xi \ge k\} \le \mathbf{E}\,\xi < 1 + \sum_{k=1}^{\infty} \mathbf{P}\{\xi \ge k\}.$$

Solution. Notice that the events $\{A_k\}$ form a partition of Ω so that $\xi = \sum_{k=1}^{\infty} \xi\, \mathbb{I}_{A_k}$. But on A_k we have $k-1 \le \xi < k$ so that we have the first set of inequalities. Now take expectation to arrive at

$$\sum_{k=1}^{\infty} (k-1)\mathbf{P}(A_k) \le \mathbf{E}\,\xi < \sum_{k=1}^{\infty} k\mathbf{P}(A_k),$$

and simplify the telescopic sum by writing $\mathbf{P}(A_k) = \mathbf{P}(\xi \ge k) - \mathbf{P}(\xi \ge k-1)$.

5. Let $\mu_n = \max\{\xi_1, \ldots, \xi_n\}$, where $\xi_1, \ldots, \xi_n$ are independent uniformly distributed on [0,1] random variables. Show that $n(1-\mu_n)$ converges weakly to the Exponentially distributed random variable with parameter 1 (i.e. with the cdf $F(x) = 1 - e^{-x}$, $x \ge 0$.
Solution. For any $y \in [0,1)$, $\mathbf{P}\{1 - \mu_n > y\} = \mathbf{P}\{\max\{\xi_1, \ldots, \xi_n\} < 1 - y\} = 1 - \mathbf{P}\big(\cap_{k=1}^{n} \{\xi_k < 1 - y\}\big) = (1-y)^n$. Therefore, for any $x \ge 0$ and any $n > x$ we will have $0 \le x/n < 1$ and thus $\mathbf{P}\{n(1-\mu_n) > x\} = (1 - x/n)^n \to e^{-x}$. The latter is $1 - F(x)$ for exponential distribution which is a continuous function. So the limit for all x means the weak convergence.