Foundations of Probability Theory (MVE140 – MSA150)

Friday 21st of December 2012 examination questions
Tid: Kl. 8.30 – 12.30

You are allowed to use a dictionary (to and from English), a university approved calculator and up to a maximum of 3 double-sided pages of your own written notes. This examination has five problems with a maximum of 20 credit points for a fully satisfactory solution, so the maximal total is 100 credit points. To pass the course, you need to score at least 40 points. The examiner, Prof. Sergei Zuyev, will be available at the examination site around 10:30am and 12pm. Telephone: 031 772 3020.

Examination Questions

1. Let $\{\mathcal{B}_\alpha : \alpha \in I\}$ be an arbitrary family of σ-fields of subsets of Ω. Show that $\cap_{\alpha\in I}\mathcal{B}_\alpha$ is a σ-algebra.

2. A bag contains a new tennis balls and b 'old' ones (i.e. already played with). A tennis coach takes two balls from the bag at random and make a training with them. At the end of the training he puts both balls back into the bag. Find the probability that at the next training session he would take out two *new* balls.

3. The joint density of random variables ξ and η is given by $f_{\xi,\eta}(x,y) = 2e^{-x-y}$ for $0 < x < y < +\infty$ and 0 otherwise.

   (a) Find the marginal densities.

   (b) Conditional density $f_{\eta|\xi=x}(y|x)$ and the conditional expectation $\mathbf{E}[\eta \mid \xi]$.

   (c) Are ξ and η independent?

4. Given n independent realisations $\xi_1, \dots, \xi_n$ of a random variable ξ with cdf $F(x)$, their arrangement in increasing order is called *variational series*: $\xi_{(1)} \le \dots \xi_{(n)}$. The kth element of this series $\xi_{(k)}$ is also called the *kth order statistic*, $k = 1, \dots, n$. Note that $\xi_{(1)} = \min\{\xi_1, \dots, \xi_n\}$ and $\xi_{(n)} = \max\{\xi_1, \dots, \xi_n\}$. Find the distributions (express it via the cdf) of

   (a) $\xi_{(1)}$ and $\xi_{(n)}$;

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(b) $\xi_{(k)}$ for a general $k = 1, \ldots, n$.

(c) In the case when ξ_i are uniform in $[0,1]$, find the conditional distribution of $\xi_{(1)}$ given $\xi_{(n)} = x$, where $x \in (0,1)$.

5. The Gamma distributed random variable $\gamma(\alpha, \beta)$ with *shape parameter* $\alpha > 0$ and *rate parameter* $\beta > 0$ has density

$$f_{\alpha,\beta}(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$$

for $x > 0$ and 0 otherwise. $\Gamma(\alpha)$ above is the Euler's Gamma-function:

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$$

which is a generalisation of the factorial: $\Gamma(n+1) = n!$ for any natural n.

(a) Compute the Laplace transform $\mathcal{L}_{\gamma(\alpha,\beta)}(z) = \mathbf{E}\, e^{-z\gamma(\alpha,\beta)}$ and the corresponding characteristic function.

(b) Show that if $\gamma(\alpha, \beta) \sim \text{Gamma}(\alpha, \beta)$, then $c\gamma(\alpha, \beta) \sim \text{Gamma}(\alpha, \beta/c)$ for any constant $c > 0$.

(c) Show that $\zeta_\alpha = (\gamma(\alpha, \beta) - \alpha)/\sqrt{\alpha}$ weakly converges when $\alpha \to \infty$ to $\mathcal{N}(0, \beta^{-2})$ Normal distribution.

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Examination Questions

1. Let $\{\mathcal{B}_\alpha : \alpha \in I\}$ be an arbitrary family of σ-fields of subsets of Ω. Show that $\cap_{\alpha\in I}\mathcal{B}_\alpha$ is a σ-algebra.
   Solution. Since $\Omega \in \mathcal{B}_\alpha$ for every α, then $\Omega \in \cap_{\alpha\in I}\mathcal{B}_\alpha$. Next, if $B \in \cap_\alpha \mathcal{B}_\alpha$ then $B \in \mathcal{B}_\alpha$ for all α, hence $B^c \in \mathcal{B}_\alpha$ for all α, i.e. $B^c \in \cap_{\alpha\in I}\mathcal{B}_\alpha$. Similarly it is shown that the countable unions are in $\cap_{\alpha\in I}\mathcal{B}_\alpha$.

2. A bag contains a new tennis balls and b 'old' ones (i.e. already played with). A tennis coach takes two balls from the bag at random and make a training with them. At the end of the training he puts both balls back into the bag. Find the probability that at the next training session he would take out two *new* balls.
   Solution. Let A_i, $i = 0, 1, 2$ be the event that i new balls out of 2 selected at the first training were new and B_2 be the event that 2 new balls are selected at the second training.

$$\mathbf{P}(A_0) = \frac{\binom{b}{2}}{\binom{a+b}{2}};\ \mathbf{P}(A_1) = \frac{ab}{\binom{a+b}{2}};\ \mathbf{P}(A_0) = \frac{\binom{a}{2}}{\binom{a+b}{2}}$$

Then $\mathbf{P}(B_2 \mid A_i) = \binom{a-i}{2}/\binom{a+b}{2}$, $i = 0, 1, 2$ because when A_i is observed, the number of new balls has become $m - i$ after the first training. Now use the total probability formula:

$$\mathbf{P}(B_2) = \frac{\binom{a}{2}}{\binom{a+b}{2}}\frac{\binom{b}{2}}{\binom{a+b}{2}} + \frac{\binom{a-1}{2}}{\binom{a+b}{2}}\frac{ab}{\binom{a+b}{2}} + \frac{\binom{a-2}{2}}{\binom{a+b}{2}}\frac{\binom{a}{2}}{\binom{a+b}{2}}$$

3. The joint density of random variables ξ and η is given by $f_{\xi,\eta}(x,y) = 2e^{-x-y}$ for $0 < x < y < +\infty$ and 0 otherwise.

   (a) Find the marginal densities.

   (b) Conditional density $f_{\eta|\xi=x}(y|x)$ and the conditional expectation $\mathbf{E}[\eta \mid \xi]$.

   (c) Are ξ and η independent?

Solution.

$$\begin{aligned} f_\xi(x) &= \int_x^\infty 2e^{-x-y}dy = 2e^{-2x}, \quad x > 0; \\ f_\eta(y) &= \int_0^y 2e^{-x-y}dx = 2e^{-y}(1-e^{-y}), \quad y > 0; \\ f_{\eta|\xi=x}(y|x) &= \frac{2e^{-x-y}}{2e^{-2x}} = e^{-(y-x)}, \quad y > x \\ \mathbf{E}[\eta \mid \xi] &= \int_\xi^\infty ye^{-(y-\xi)}dy = \xi + 1. \end{aligned}$$

The variables are dependent since $f_{\xi,\eta}(x,y) \neq f_\xi(x)f_\eta(y)$.

4. Given n independent realisations $\xi_1, \ldots, \xi_n$ of a random variable ξ with cdf $F(x)$, their arrangement in increasing order is called *variational series*: $\xi_{(1)} \leq \ldots \xi_{(n)}$. The kth element of this series $\xi_{(k)}$ is also called the *kth order statistic*, $k = 1, \ldots, n$. Note that $\xi_{(1)} = \min\{\xi_1, \ldots, \xi_n\}$ and $\xi_{(n)} = \max\{\xi_1, \ldots, \xi_n\}$. Find the distributions (express it via the cdf) of

   (a) $\xi_{(1)}$ and $\xi_{(n)}$;

   (b) $\xi_{(k)}$ for a general $k = 1, \ldots, n$.

   (c) In the case when ξ_i are uniform in $[0,1]$, find the conditional distribution of $\xi_{(1)}$ given $\xi_{(n)} = x$, where $x \in (0,1)$.

Solution. $\xi_{(k)} \le x$ when exactly k realisations do not exceed x and the rest are larger than x. This corresponds to k successes in n Bernoulli trials with success probability $\mathbf{P}\{\xi \le x\} = F(x)$ so that

$$\mathbf{P}\{\xi_{(k)} \le x\} = \binom{n}{k} F(x)^k (1 - F(x))^{n-k}.$$

When $\xi \sim \mathrm{Unif}[0,1]$, $\mathbf{P}\{\xi_{(n)} \le x\} = x^n$ for $x \in [0,1]$ so that the pdf of $\xi_{(n)}$ is $nx^{n-1}\,\mathbb{I}_{[0,1]}(x)$. For any $0 \le y < x$, $\mathbf{P}\{\xi_{(1)} > y \mid \xi_{(n)} = x;\ \xi_1 = \xi_{(n)}\} = (x-y)^{n-1}$, because all $n-1$ values $\xi_2, \dots, \xi_n$ must lie in $[y, x]$. Obviously, the same probability is for condition $\xi_k = \xi_{(n)}$. Writing the full probability formula and using that $\mathbf{P}\{\xi_k = \xi_{(n)}\} = 1/n$ for all k, we have that $\mathbf{P}\{\xi_{(1)} > y \mid \xi_{(n)} = x\} = (x-y)^{n-1}$ with the pdf

$$f_{\xi_{(1)}|\xi_{(n)}=x}(y) = (n-1)(x-y)^{n-2}\,\mathbb{I}_{[0,x]}(y).$$

5. The Gamma distributed random variable $\gamma(\alpha, \beta)$ with *shape parameter* $\alpha > 0$ and *rate parameter* $\beta > 0$ has density

$$f_{\alpha,\beta}(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$$

for $x > 0$ and 0 otherwise. $\Gamma(\alpha)$ above is the Euler's Gamma-function:

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$$

which is a generalisation of the factorial: $\Gamma(n+1) = n!$ for any natural n.

(a) Compute the Laplace transform $\mathcal{L}_{\gamma(\alpha,\beta)}(z) = \mathbf{E}\, e^{-z\gamma(\alpha,\beta)}$ and the corresponding characteristic function.

(b) Show that if $\gamma(\alpha, \beta) \sim \mathrm{Gamma}(\alpha, \beta)$, then $c\gamma(\alpha, \beta) \sim \mathrm{Gamma}(\alpha, \beta/c)$ for any constant $c > 0$.

(c) Show that $\zeta_\alpha = (\gamma(\alpha, \beta) - \alpha)/\sqrt{\alpha}$ weakly converges when $\alpha \to \infty$ to $\mathcal{N}(0, \beta^{-2})$ Normal distribution.

Solution.

$$\mathcal{L}_{\gamma(\alpha,\beta)}(z) = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty e^{-zx} x^{\alpha-1} e^{-\beta x} dx$$
$$= \frac{\beta^\alpha}{(\beta+x)^\alpha} \Big[\frac{(\beta+z)^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{-(\beta+z)x} dx \Big] = \Big(1 + \frac{z}{\beta}\Big)^{-\alpha}$$

since the expression in the square brakets is the density of Gamma$(\alpha, \beta + z)$ distribution which integrates to 1. Thus the characteristic function is

$$\varphi_{\gamma(\alpha,\beta)}(t) = \mathcal{L}_{\gamma(\alpha,\beta)}(-it) = \Big(1 - \frac{it}{\beta}\Big)^{-\alpha}.$$

To show (b), just verify that $\mathcal{L}_{c\gamma(\alpha,\beta)}(z) = \mathcal{L}_{\gamma(\alpha,\beta/c)}$.
For (c), expand the characteristic function

$$\varphi_{\zeta_\alpha}(t) = e^{-it\sqrt{\alpha}} \varphi_{\gamma(\alpha,\beta)}(t/\sqrt{\alpha}) = \exp\Big\{-it\sqrt{\alpha} - \alpha \log\Big(1 - \frac{it}{\sqrt{\alpha}\beta}\Big)\Big\}$$

at $t = 0$ to see that $\varphi_{\zeta_\alpha}(t) = \exp\{-t^2/(2\beta^2) + o(\alpha^{-1})\}$ which converges to the characteristic function of $\mathcal{N}(0, \beta^{-2})$.