Foundations of Probability Theory (MVE140 – MSA150)

Thursday 13th of January 2012 resit examination questions

You are allowed to use a dictionary (to and from English) and up to a maximum of 5 double-sided pages of your own written notes. This examination has five problems with a maximum of 20 credit points for a fully satisfactory solution, so the maximal total is 100 credit points. To pass the course, you need to score at least 40 points. An assistant of the examiner, Anton Muratov, (5377) ***is available around 10:30 and 12:00 for questions concerning the problems' formulation.***

1. A poker player receives 5 cards from a standard card deck containing 52 cards. What is the probability that he gets 4 cards of a kind, e.g. 4 aces or 4 kings, etc.? Do you think that the player is cheating if he shows twice such a situation in five consecutive games? (Take 0.001% as the cutoff point for your belief, so that you would not believe that the game is fair if you observe a too rare event, i.e. the probability of which is below 0.001%).

2. Two symmetric dice are rolled. Find the expected sum of the shown numbers under the condition that two *different* numbers are observed.

3. Let U be a uniformly distributed in $[0, 1]$ random variable. Find the density (and carefully identify the region where it is non-zero) of a random variable $\xi = U^{1/\alpha}$ for

   a) $\alpha > 0$; and

   b) $\alpha < 0$.

   For which range of α the random variable ξ is integrable, i.e. $\mathbf{E}\,\xi < \infty$?

4. $n \geq 2$ points $U_1, \ldots, U_n$ are thrown uniformly and independently on $[0, 1]$. Denote $\eta_1 = \min_{1 \leq k \leq n} U_k$ and $\eta_2 = \max_{1 \leq k \leq n} U_k$. Find:

   a) The c.d.f. of η_1;

   b) The density of η_2;

PLEASE, TURN PAGE OVER!

c) The joint distribution of the pair (η_1, η_2);

d) The conditional density of η_2 given $\eta_1 = x$, $x \in [0,1]$.

5. ξ_n is a random variable taking values $-n$ and n with equal probabilities. Do the following sequences have a weak limit?

   a) $\xi_n/\sqrt{n}$?

   b) $\xi_n/\sqrt{\mathbf{var}\,\xi_n}$?

Lösningar

Foundations of Probability Theory
(MVE140 – MSA150)

Thursday 13th of January 2012 resit examination questions

You are allowed to use a dictionary (to and from English) and up to a maximum of 5 double-sided pages of your own written notes. This examination has five problems with a maximum of 20 credit points for a fully satisfactory solution, so the maximal total is 100 credit points. To pass the course, you need to score at least 40 points. An assistant of the examiner, Anton Muratov, is available around 10:30 and 12:00 for questions concerning the problems' formulation.

1. A poker player receives 5 cards from a standard card deck containing 52 cards. What is the probability that he gets 4 cards of a kind, e.g. 4 aces or 4 kings, etc.? Do you think that the player is cheating if he shows twice such a situation in five consecutive games? (Take 0.001% as the cutoff point for your belief, so that you would not believe that the game is fair if you observe a too rare event, i.e. the probability of which is below 0.001%).
   Solution. 13 variants to choose the kind and 48 variants for the other card. So $p = 13 \cdot 48/\binom{52}{5} = 0.00024$. Probability of having twice such a situation in 5 games is binomial: $\binom{5}{2}p^2(1-p)^{5-2} = 5.76 \cdot 10^{-7} < 10^{-5}$ so the player must be cheating.

2. Two symmetric dice are rolled. Find the expected sum of the shown numbers under the condition that two *different* numbers are observed.
   Solution. Conditional distribution of the sum ξ:

ξ	3	4	5	6	7	8	9	10	11
P	2/30	2/30	4/30	4/30	6/30	4/30	4/30	2/30	2/30

   Hence, $\mathbf{E}[\xi \mid \text{different numbers}] = 7$.

3. Let U be a uniformly distributed in $[0,1]$ random variable. Find the density (and carefully identify the region where it is non-zero) of a random variable $\xi = U^{1/\alpha}$ for

   a) $\alpha > 0$; and

   b) $\alpha < 0$.

   For which range of α the random variable ξ is integrable, i.e. $\mathbf{E}\xi < \infty$?
   Solution. The p.d.f. is $f_\xi(x) = \alpha x^{\alpha-1}\, \mathbb{I}_{[0,1]}(x)$ if $\alpha > 0$ and $f_\xi(x) = |\alpha| x^{\alpha-1}\, \mathbb{I}_{[1,\infty]}(x)$ if $\alpha < 0$. $\mathbf{E}\xi < \infty$ if $\alpha < -1$ or $\alpha > 0$.

4. $n \geq 2$ points $U_1, \ldots, U_n$ are thrown uniformly and independently on $[0,1]$. Denote $\eta_1 = \min_{1\le k\le n} U_k$ and $\eta_2 = \max_{1\le k\le n} U_k$. Find:

   a) The c.d.f. of η_1;

   b) The density of η_2;

   c) The joint distribution of the pair (η_1, η_2);

   d) The conditional density of η_2 given $\eta_1 = x$, $x \in [0,1]$.

Solution. a) $\eta_1 > x$, for $x \in [0,1]$ when all the points u_k are inside $[x,1]$, so by the independence, $F_{\eta_1}(x) = 1 - (1-x)^n$. Obviously, it is 0 for $x < 0$ and 1 for $x > 1$.
b) Similarly, $F_{\eta_2}(x) = x^n$ since $\eta_2 \le x$ iff all $u_k \le x$. Thus $f_{\eta_2}(x) = nx^{n-1}$.
c) When $x > y$, $\mathbf{P}\{\eta_1 \le x;\ \eta_2 \le y\} = \mathbf{P}\{\eta_2 \le y\} = F_{\eta_2}(y)$. For $0 \le x \le y \le 1$, $\mathbf{P}\{x < \eta_1 \le \eta_2 \le y\} = (y-x)^n$, so that

$$F_{(\eta_1,\eta_2)}(x,y) = \mathbf{P}\{\eta_2 \le y\} - \mathbf{P}\{x < \eta_1 \le \eta_2 \le y\} = y^n - (y-x)^n\,.$$

So the p.d.f. is

$$f_{(\eta_1,\eta_2)}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{(\eta_1,\eta_2)}(x,y) = n(n-1)(y-x)^{n-2}$$

when $0 \le x \le y \le 1$ and 0 otherwise.
d)

$$f_{\eta_2|\eta_1=x}(y) = \frac{f_{(\eta_1,\eta_2)}(x,y)}{f_{\eta_1}(x)} = \frac{(n-1)(y-x)^{n-2}}{(1-x)^{n-1}}$$

when $0 \le x \le y \le 1$ and 0 otherwise.

5. ξ_n is a random variable taking values $-n$ and n with equal probabilities. Do the following sequences have a weak limit?

   a) $\xi_n/\sqrt{n}$?

   b) $\xi_n/\sqrt{\mathbf{var}\,\xi_n}$?

Solution. The characteristic function: $\varphi_{\xi_n}(t) = 0.5e^{-itn} + 0.5e^{itn} = \cos(tn)$. a) $\varphi_{\xi_n/\sqrt{n}}(t) = \cos(t\sqrt{n})$ has no limit as $n \to \infty$ for any $t \ne 0$, so no weak limit in this case. b) $\mathbf{var}\,\xi_n = n^2$, $\varphi_{\xi_n/n}(t) = \cos(t)$, a continuous function in 0, so the weak limit exists and equals the distribution of a r.v. taking values ± 1 with equal probabilities (actually, all ξ_n/n are distributed like this, hence is the limit).