Foundations of Probability Theory (MVE140 – MSA150)

Friday 16th of December 2011 examination questions

You are allowed to use a dictionary (to and from English) and up to a maximum of 5 double-sided pages of your own written notes. This examination has five problems with a maximum of 20 credit points for a fully satisfactory solution, so the maximal total is 100 credit points. To pass the course, you need to score at least 40 points. The examiner, Prof. Sergei Zuyev, is available at the examination site around 10:30am and 12pm. Telephone: 031 772 3020.

Examination Questions

1. Let $\xi : \Omega \mapsto \mathbb{R}$ be an arbitrary mapping. Denote by $\sigma(\xi)$ the following collection of subsets of Ω: $\{\xi^{-1}(B) : B \in \mathcal{B}\}$, where B runs through all Borel subsets $\mathcal{B}$ of $\mathbb{R}$. Show that $\sigma(\xi)$ is a σ-algebra[1]. Find $\sigma(\xi)$ for a mapping ξ taking just two distinct values. When $\sigma(\xi)$ for such ξ coincides with the set 2^Ω of all subsets of Ω?

2. In a certain population, the probability that the first child born to a woman is a boy is 0.51. It was also noticed that the probability that the second child is of the same sex as the first one is 0.55. The second child of a randomly selected woman is a girl. What is the probability that her first child is a boy?

3. Let ξ_1, ξ_2 be two Exponentially distributed r.v.'s, their c.d.f.'s are $F_{\xi_1}(x) = 1 - e^{-\lambda_1 x}$ and $F_{\xi_1}(x) = 1 - e^{-\lambda_2 x}$, respectively, for some $\lambda_1, \lambda_2 > 0$. Find:

   (a) the distribution of $\eta_1 = \min\{\xi_1, \xi_2\}$;

   (b) the distribution of $\eta_2 = \max\{\xi_1, \xi_2\}$;

   (c) the joint distribution of η_2 and η_1 in the case when $\lambda_1 = \lambda_2 = \lambda$;

   (d) the conditional distribution of η_2 given η_1 in the case when $\lambda_1 = \lambda_2 = \lambda$.

4. Suppose that the amounts R_n you win in n-th game of chance are independent identically distributed random variables with a finite mean m and variance σ^2. It is reasonable to assume that $m < 0$. Show that $\mathbf{P}\{(R_1 + \cdots + R_n)/n < m/2\} \to 1$ as $n \to \infty$. What is the moral of this result?

5. Let η_n be Poisson distributed random variables with parameter n. Does there exist a weak limit of the sequence $(\eta_n - n)\sqrt{n}$ and if is does, what is it?

[1] It is called the σ-algebra *generated by* ξ.

Lösningar

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1. Let $\xi : \Omega \mapsto \mathbb{R}$ be an arbitrary mapping. Denote by $\sigma(\xi)$ the following collection of subsets of Ω: $\{\xi^{-1}(B) : B \in \mathcal{B}\}$, where B runs through all Borel subsets $\mathcal{B}$ of $\mathbb{R}$. Show that $\sigma(\xi)$ is a σ-algebra[1]. Find $\sigma(\xi)$ for a mapping ξ taking just two distinct values. When $\sigma(\xi)$ for such ξ coincides with the set 2^Ω of all subsets of Ω?
Solution. For any $B_1, B_2, \dots \in \mathcal{B}$, $\xi^{-1}(B_1) \cap \xi^{-1}(B_2) = \{\omega : \xi(\omega) \in B_1 \text{ and } \xi(\omega) \in B_2 \text{ and} \dots\} = \xi^{-1}(\cap_i B_i) \in \sigma(\xi)$ since $\cap_i B_i$ is also in $\mathcal{B}$. Also, for any $B \in \mathcal{B}$, $(\xi^{-1}(B))^{\mathrm{c}} = \{\omega : \xi(\omega) \in B^{\mathrm{c}}\} = \xi^{-1}(B^{\mathrm{c}}) \in \sigma(\xi)$. When $\xi(\omega) = c_1$ on $S \subset \Omega$ and hence $\xi(\omega) = c_2$ on $\Omega \setminus S$, then $\xi^{-1}(B) = S$, $\Omega \setminus S = S^{\mathrm{c}}$, Ω or $\emptyset$, depending on whether $c_1 \in B$, $c_2 \in B$, both or none. So $\sigma(\xi) = \{\emptyset, S, S^{\mathrm{c}}, \Omega\}$. This is 2^Ω if Ω itself consists of 2 elements, so that S and S^{c} are one-point sets.

2. In a certain population, the probability that the first child born to a woman is a boy is 0.51. It was also noticed that the probability that the second child is of the same sex as the first one is 0.55. The second child of a randomly selected woman is a girl. What is the probability that her first child is a boy?
Solution. Let B_1 be the event that the first child is a boy and B_2 is that the second child is a boy. By the Bayes formula, $\mathbf{P}(B_1 \mid B_2{}^{\mathrm{c}}) = \mathbf{P}(B_2{}^{\mathrm{c}} \mid B_1)\mathbf{P}(B_1)/(\mathbf{P}(B_2{}^{\mathrm{c}} \mid B_1)\mathbf{P}(B_1) + \mathbf{P}(B_2{}^{\mathrm{c}} \mid B_1{}^{\mathrm{c}})\mathbf{P}(B_1{}^{\mathrm{c}})) = 0.45 \cdot 0.51/(0.45 \cdot 0.51 + 0.55 \cdot 0.49) = 0.4599$.

3. Let ξ_1, ξ_2 be two independent Exponentially distributed r.v.'s, their c.d.f.'s are $F_{\xi_1}(x) = 1 - e^{-\lambda_1 x}$ and $F_{\xi_1}(x) = 1 - e^{-\lambda_2 x}$, respectively, for some $\lambda_1, \lambda_2 > 0$. Find:
   (a) the distribution of $\eta_1 = \min\{\xi_1, \xi_2\}$;
   (b) the distribution of $\eta_2 = \max\{\xi_1, \xi_2\}$;
   (c) the joint distribution of η_2 and η_1 in the case when $\lambda_1 = \lambda_2 = \lambda$;
   (d) the conditional distribution of η_2 given η_1 in the case when $\lambda_1 = \lambda_2 = \lambda$.

[1] It is called the σ-algebra *generated by* ξ.

Solution.

(a) $\mathbf{P}\{\eta_1 > x\} = \mathbf{P}\{\xi_1 > x; \xi_2 > x\} = e^{-\lambda_1} e^{-\lambda_2 x} = e^{-(\lambda_1+\lambda_2)x}$, so that $\eta_1 \sim \text{Exp}(\lambda_1 + \lambda_2)$.

(b) $\mathbf{P}\{\eta_2 \le x\} = \mathbf{P}\{\xi_1 \le x; \xi_2 \le x\} = (1 - e^{-\lambda_1})(1 - e^{-\lambda_2 x})$

(c) If $x > y$ then $\mathbf{P}\{\eta_1 \le x;\ \eta_2 \le y\} = \mathbf{P}\{\eta_2 \le y\}$ which is $F_{\eta_2}(y)$ above. Consider now the case $x \le y$. Since ξ_1 and ξ_2 are independent, the pair (ξ_1, ξ_2) has joint cdf $F_{(\xi_1,\xi_2)}(x,y) = (1 - e^{-\lambda x})(1 - \varepsilon^{-\lambda y})$ for $x, y \ge 0$ and 0 otherwise. The set $\{(z_1, z_2) \in \mathbb{R}^2_+ : \min\{z_1, z_2\} \le x;\ \max\{z_1, z_2\} \le y\}$ is the union of two rectangles, each having a vertex at the origin and the opposite vertex in the point (x, y) and (y, x), respectively. By the symmetry, the measure of each of them is $F_{(\xi_1,\xi_2)}(x, y)$, the measure of their intersection is $F_{(\xi_1,\xi_2)}(x, x)$. Hence

$$\mathbf{P}\{\eta_1 \le x;\ \eta_2 \le y\} = 2F_{(\xi_1,\xi_2)}(x,y) - F_{(\xi_1,\xi_2)}(x,x) = 1 - e^{-2\lambda x} - 2e^{-\lambda y} + 2e^{-\lambda(x+y)}.$$

(d) Obviously, $\mathbf{P}\{\eta_2 > y \mid \eta_1 = x\} = 1$ for all $y < x$. Consider $y \ge x$ and put $z = y - x > 0$. Since ξ_1 and ξ_2 are equally distributed,

$$\begin{aligned}\mathbf{P}\{\eta_2 > y \mid \eta_1 = x\} &= \mathbf{P}\{\xi_2 > y \mid \xi_1 = x; \xi_2 > x\}\mathbf{P}\{\xi_1 < \xi_2\} \\ &\quad + \mathbf{P}\{\xi_1 > y \mid \xi_2 = x; \xi_1 > x\}\mathbf{P}\{\xi_1 > \xi_2\} \\ &= e^{-\lambda y}/e^{-\lambda x} \cdot 0.5 + e^{-\lambda y}/e^{-\lambda x} \cdot 0.5 = e^{-\lambda z}\end{aligned}$$

which corresponds to the distribution of a r.v. $\eta_1 + \xi$, where $\xi \sim \text{Exp}(\lambda)$.

4. Suppose that the amounts R_n you win in n-th game of chance are independent identically distributed random variables with a finite mean m and variance σ^2. It is reasonable to assume that $m < 0$. Show that $\mathbf{P}\{(R_1 + \cdots + R_n)/n < m/2\} \to 1$ as $n \to \infty$. What is the moral of this result?
Solution. Let $S_n = R_1 + \ldots R_n$. Since $m < 0$ and $\mathbf{E}\, S_n/n = m$, we have that

$$\mathbf{P}\{S_n/n > m/2\} = \mathbf{P}\{S_n/n - m > -m/2\} = \mathbf{P}\{S_n/n - m > |m|/2\}.$$

Now by the Chebyshov inequality, the later is at most

$$\mathbf{P}\{|S_n/n - m| > |m|/2\} \le \mathbf{var}(S_n/n)/(m^2/4) = 4\sigma^2/(nm^2) \to 0 \text{ as } n \to \infty.$$

5. Let η_n be Poisson distributed random variables with parameter n. Does there exist a weak limit of the sequence $(\eta_n - n)\sqrt{n}$ and if is does, what is it?

Solution. If there is a limit, then the characteristic function of $\xi_n = (\eta_n - n)\sqrt{n}$ converges for all values of its argument to a continuous function at 0. Wwe have by the Shift theorem that

$$\varphi_{\xi_n}(t) = \exp\{-in\sqrt{n}t + n(e^{it\sqrt{n}} - 1)\},$$

so whatever is $t \neq 0$, there is no limit of $\varphi_{\xi_n}(t)$ so that there is no weak limit either.