Foundations of Probability Theory (MVE140 – MSA150)

Thursday 13th of January 2011 resit examination questions

You are allowed to use a dictionary (to and from English) and up to a maximum of 5 pages of your own written notes. This examination has five problems with a maximum of 20 credit points for a fully satisfactory solution, so the maximal total is 100 credit points. To pass the course, you need to score at least 40 points. The examiner, Prof. Sergei Zuyev, is available at the examination site around 10:30am and 12:30pm. Telephone: 076 014 8990.

1. A fair die is rolled twice. Define the following events:

$$A = \{\text{the first roll shows an odd number}\}$$
$$B = \{\text{the second roll shows an odd number}\}$$
$$A = \{\text{the sum of the two numbers obtained is odd}\}.$$

Are the events

a) Pairwise independent?

b) Mutually independent?

2. Let ξ be a random variable taking only non-negative integer values. Show that $\mathbf{E}\,\xi = \sum_{n=1}^{\infty} \mathbf{P}\{\xi \geq n\}$ whenever this is a finite number or infinity.

3. A wire of length 1m is cut into two pieces at an arbitrary point. One piece is bent into a square, the other piece - into a circle. Find the probability that the area of the square is larger than the area of the circle.

4. Let $\xi_1, \xi_2, \dots$ be a sequence of independent random variables and ν be positive integer valued random variable independent of $\xi_n'{s}$. Prove the following *Wald identity*: $\mathbf{E}\sum_{n=1}^{\nu} \xi_n = \mathbf{E}\,\xi_1\,\mathbf{E}\,\nu$ (the number of terms in the sum is random, so you cannot just write that the expectation of the sum is sum of expectations!)

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5. Prove the following theorem due to Rényi: assume ξ_n are Geometrically distributed with parameters p_n (i.e. $\mathbf{P}\{\xi_n = k\} = p_n(1-p_n)^{k-1},\ k = 1, 2, \ldots$), where $\lim_{n\to\infty} p_n = 0$. Then the sequence $\{p_n \xi_n\}$ converges weakly to an exponentially distributed random variable ξ with parameter 1 (i.e. $\mathbf{P}\{\xi > x\} = e^{-x},\ x \geq 0$).

LÖSNINGAR

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1. A fair die is rolled twice. Define the following events:

$$A = \{\text{the first roll shows an odd number}\}$$
$$B = \{\text{the second roll shows an odd number}\}$$
$$C = \{\text{the sum of the two numbers obtained is odd}\}.$$

Are the events

a) Pairwise independent?

b) Mutually independent?

Solution. There are 36 equiprobable elementary outcomes. There are 18 possibilities for A even to occur, so $\mathbf{P}(A) = 18/36 = 1/2$. Similarly, $\mathbf{P}(B) = 1/2 = \mathbf{P}(C)$. a) - Answer: Yes (check that $\mathbf{P}(AB) = \mathbf{P}(A)\mathbf{P}(B)$, $\mathbf{P}(AC) = \mathbf{P}(A)\mathbf{P}(C)$ and $\mathbf{P}(BC) = \mathbf{P}(B)\mathbf{P}(C)$). b) - No, because $\mathbf{P}(ABC) = 0 \neq 1/2^3 = 1/8$.

2. Let ξ be a random variable taking only non-negative integer values. Show that $\mathbf{E}\,\xi = \sum_{n=1}^{\infty} \mathbf{P}\{\xi \geq n\}$ whenever this is a finite number or infinity.

Solution. Denote $p_n = \mathbf{P}\{\xi = n\}$, $n = 0, 1, 2, \ldots$. Then

$$\sum_{n=1}^{\infty} \mathbf{P}\{\xi \geq n\} = (p_1+p_2+p_3+p_4+\ldots)+(p_2+p_3+p_4+\ldots)+(p_3+p_4+\ldots)+\ldots$$

$$= p_1 + 2p_2 + 3p_3 + \cdots = \sum_{n=1}^{\infty} np_n = \mathbf{E}\,\xi\,.$$

The change of summation order is legal since a positive convergent series is also absolutely convergent.

3. A wire of length 1m is cut into two pieces at an arbitrary point. One piece is bent into a square, the other piece - into a circle. Find the probability that the area of the square is larger than the area of the circle.

   Solution. Let U be the length of the first piece. It is uniformly distributed in $(0, 1)$. Then

$$P = \mathbf{P}\Big\{\Big(\frac{U}{4}\Big)^2 \geq \pi\Big(\frac{1-U}{2\pi}\Big)^2\Big\}$$
$$= \mathbf{P}\{(4-\pi)U^2 - 8U + 4 < 0\} = \mathbf{P}\{x_1 < U < x_2\}\,,$$

   where

$$x_1 = \frac{8 - \sqrt{64 - 16(4-\pi)}}{2(4-\pi)} \approx 0.53\,, \quad x_2 = \frac{8 + \sqrt{64 - 16(4-\pi)}}{2(4-\pi)} \approx 8.79\,.$$

   So $P = \mathbf{P}\{U > x_1\} = 1 - x_1 \approx 1 - 0.53 = 0.47$.

4. Let $\xi_1, \xi_2, \ldots$ be a sequence of independent random variables and ν be positive integer valued random variable independent of ξ_n's. Prove the following *Wald identity*: $\mathbf{E}\sum_{n=1}^{\nu} \xi_n = \mathbf{E}\,\xi_1\,\mathbf{E}\,\nu$ (the number of terms in the sum is random, so you cannot just write that the expectation of the sum is sum of expectations!)

Solution.

$$\begin{aligned}\mathbf{E}\sum_{n=1}^{\nu}\xi_n &= \sum_{k=1}^{\infty}\mathbf{E}\Big[\sum_{n=1}^{k}\xi_n \;\Big|\; \nu=k\Big]\,\mathbf{P}\{\nu=k\}\\ &= \sum_{k=1}^{\infty}\sum_{n=1}^{k}\mathbf{E}[\xi_n \mid \nu=k]\,\mathbf{P}\{\nu=k\} = \sum_{k=1}^{\infty}\sum_{n=1}^{k}\mathbf{E}\,\xi_1\,\mathbf{P}\{\nu=k\}\\ &= \sum_{k=1}^{\infty}k\,\mathbf{E}\,\xi_1\,\mathbf{P}\{\nu=k\} = \mathbf{E}\,\xi_1\sum_{k=1}^{\infty}k\,\mathbf{P}\{\nu=k\} = \mathbf{E}\,\xi_1\,\mathbf{E}\,\nu\,.\end{aligned}$$

The third identity is due to independence of ξ_i's and ν and identical distributions of ξ_i's, so that $\mathbf{E}[\xi_n \mid \nu=k] = \mathbf{E}\,\xi_n = \mathbf{E}\,\xi_1$.

5. Prove the following theorem due to Rényi: assume ξ_n are Geometrically distributed with parameters p_n (i.e. $\mathbf{P}\{\xi_n=k\} = p_n(1-p_n)^{k-1},\ k = 1,2,\dots$), where $\lim_{n\to\infty} p_n = 0$. Then the sequence $\{p_n\xi_n\}$ converges weakly to an exponentially distributed random variable ξ with parameter 1 (i.e. $\mathbf{P}\{\xi > x\} = e^{-x},\ x \geq 0$).

Solution. Since $\mathbf{P}\{p_n\xi_n > t\} = \sum_{k=[t/p_n]+1}^{\infty}(1-p_n)p_n^k = (1-p_n)^{[t/p_n]+1} \to e^{-t}$ since $\lim_n(1+t/n)^n = e^{-t}$. The latter is $\mathbf{P}\{\xi > t\}$ for Exp(1)-distributed ξ – a continuous function at every $t > 0$.