Foundations of Probability Theory
(MVE140 – MSA150)

Friday 17th of December 2010 examination questions

You are allowed to use a dictionary (to and from English) and up to a maximum of 5 pages of your own written notes. This examination has five problems with a maximum of 20 credit points for a fully satisfactory solution, so the maximal total is 100 credit points. To pass the course, you need to score at least 40 points. The examiner, Prof. Sergei Zuyev, is available at the examination site around 10:30am and 12pm. Telephone: 031 772 3020.

Examination Questions

1. Telegraphic signals *dash* and *dot* are sent in the proportion 4:3. Due to weather conditions causing very erratic transmission a *dot* becomes a *dash* with probability 1/4, whereas a *dash* becomes *dot* with probability 1/3. If a *dot* is received, that is the probability that it actually a *dot* that has been sent?

2. Jack cuts off a piece from 1m length wire at an arbitrary place and takes it with him. The piece left is picked up by John who cuts off a piece from it at arbitrary place and throws away the remainder. Find the distribution of the length of wire John has. Find the covariance between the lengths of wires Jack and John have. Are these lengths independent?

3. Gamma distribution $\Gamma(n, \lambda)$ is the distribution of a sum of n independent exponentially $\mathrm{Exp}(\lambda)$ distributed random variables. Find

   (a) its characteristic function;

   (b) its mean and its variance;

   (c) For $a > 0$ find the weak limit of the sequence $\{\zeta_n = \xi_n - \sqrt{n}/a\}$, where ξ_n are random variables distributed as $\Gamma(n, a\sqrt{n})$.

4. Let $\xi_1, \xi_2, \dots$ be a sequence of independent Bernoulli-distributed with parameter p random variables and ν be a Poisson distributed random variable with parameter λ independent of ξ_n's. Find the characteristic

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function of the sum $\zeta = \sum_{n=1}^{\nu} \xi_n$ (the sum is understood as 0, if $\nu = 0$). Find the expectation and the variance of ζ.

5. Prove the 'memoryless' property of the Geometric distribution: if $\xi \sim$ Geom(p), i.e. $\mathbf{P}\{\xi_n = k\} = p_n(1-p_n)^{k-1}$, $k = 1, 2, \ldots$, then the conditional distribution of $\xi - k$ given $\xi > k$ is the same as the distribution of ξ for any $k = 1, 2, \ldots$. **NB.** You may think of this property this way: time to wait until the first Head in a series of a coin tosses has the same distribution no matter how many Tails you have already observed.

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1. Telegraphic signals *dash* and *dot* are sent in the proportion 4:3. Due to weather conditions causing very erratic transmission a *dot* becomes a *dash* with probability 1/4, whereas a *dash* becomes *dot* with probability 1/3. If a *dot* is received, that is the probability that it actually a *dot* that has been sent?
   Solution. Bayes theorem: answer 27/43

2. Jack cuts off a piece from 1m length wire at an arbitrary place and takes it with him. The piece left is picked up by John who cuts off a piece from it at arbitrary place and throws away the remainder. Find the distribution of the length of wire John has. Find the covariance between the lengths of wires Jack and John have. Are these lengths independent?

Solution. Let $U_1, U_2 \sim [0,1]$ i.i.d. Then the length Jack has is distributed as U_1 and for John it is $U_2(1-U_1)$. Since $1-U_1 \sim \text{Unif}[0,1]$, then for $0 \le t \le 1$

$$\mathbf{P}\{U_2(1-U_1) \le t\} = \mathbf{P}\{U_1 U_2 \le t\}$$
$$= \mathbf{P}\{U_1 \le t\} + \int_t^1 \mathbf{P}\{U_2 \le t/x\} f_{U_1}(x)dx = t + \int_t^1 t/x\, dx = t - t\log t\,.$$

Next,

$$\mathbf{cov}(U_1, U_2(1-U_1)) = \mathbf{cov}(U_1, U_2) - \mathbf{cov}(U_1, U_1U_2)$$
$$= 0 - \mathbf{E}\, U_1^2 U_2 + \mathbf{E}\, U_1\, \mathbf{E}\, U_1 U_2 = -1/2 \int_0^1 x^2 dx + (1/2)^3 = -1/6 + 1/8 = -1/24$$

3. Gamma distribution $\Gamma(n, \lambda)$ is the distribution of a sum of n independent exponentially $\text{Exp}(\lambda)$ distributed random variables. Find
   (a) its characteristic function;
   (b) its mean and its variance;
   (c) For $a > 0$ find the weak limit of the sequence $\{\zeta_n = \xi_n - \sqrt{n}/a\}$, where ξ_n are random variables distributed as $\Gamma(n, a\sqrt{n})$.

Solution. For $\xi_1 \sim \text{Exp}(\lambda)$ the ch.f. is $\varphi_1(t) = \lambda/(\lambda - it)$, with mean $1/\lambda$ and variance $1/\lambda^2$ (e.g., by differentiating the ch.f. at $t=0$: $\mathbf{E}\,\xi^k = i^k \varphi_1^{(k)}(0)$). Hence for Gamma-distributed ξ, $\varphi_\xi(t) = \lambda^n/(\lambda - it)^n$, $\mathbf{E}\,\xi = -i\varphi_\xi'(0) = n/\lambda$, $\mathbf{var}\,\xi = n/\lambda^2$ as the sum of n independent exponentially distributed r.v.'s. Now, either use CLT for this sum: its mean is $\sqrt{n}/a$ and variance is $1/a^2$, so $a\zeta_n \xrightarrow{w} \mathcal{N}(0,1)$ implying $\zeta_n \xrightarrow{w} \mathcal{N}(0, a^{-2})$. Or show directly that $\varphi_{\zeta_n}(t) = e^{-it\sqrt{n}/a} a^n n^{n/2}/(a\sqrt{n} - it)^n \to e^{-t^2/(2a^2)}$ for all $t \in \mathbb{R}$.

4. Let $\xi_1, \xi_2, \ldots$ be a sequence of independent Bernoulli-distributed with parameter p random variables and ν be a Poisson distributed random variable with parameter λ independent of ξ_n's. Find the characteristic function of the sum $\zeta = \sum_{n=1}^{\nu} \xi_n$ (the sum is understood as 0, if $\nu = 0$).

Find the expectation and the variance of ζ.
Solution. $\varphi_\xi(t) = 1 - p + pe^{it}$,

$$\varphi_\zeta(t) = \sum_{n=0}^{\infty} (\mathbf{E}\, e^{it\xi})^n \lambda^n/(n!)\, e^{-\lambda} = \sum_{n=0}^{\infty} (\varphi_\xi(t)\lambda)^n/(n!)\, e^{-\lambda} = e^{\lambda(\varphi_\xi(t)-1)}$$
$$= e^{\lambda p(e^{it}-1)} = 1 + it\lambda p - \frac{t^2}{2}(\lambda p + \lambda^2 p^2) + o(t^2)\,.$$

Hence $\mathbf{E}\,\zeta = \lambda p$, $\mathbf{E}\,\zeta^2 = \lambda p + \lambda^2 p^2$ and $\mathbf{var}\,\zeta = \lambda p$ also.

5. Prove the 'memoryless' property of the Geometric distribution: if $\xi \sim$ Geom(p), i.e. $\mathbf{P}\{\xi_n = k\} = p_n(1-p_n)^{k-1}$, $k = 1, 2, \ldots$, then the conditional distribution of $\xi - k$ given $\xi > k$ is the same as the distribution of ξ for any $k = 1, 2, \ldots$. **NB.** You may think of this property this way: time to wait until the first Head in a series of a coin tosses has the same distribution no matter how many Tails you have already observed.
Solution.

$$\mathbf{P}\{\xi > k + n \mid \xi > k\} = \frac{\mathbf{P}\{\xi > k+n,\ \xi > k\}}{\mathbf{P}\{\xi > k\}} = \frac{\mathbf{P}\{\xi > k+n\}}{\mathbf{P}\{\xi > k\}}$$
$$= \frac{(1-p)^{k+n}}{(1-p)^k} = (1-p)^n = \mathbf{P}\{\xi > n\}\,.$$