Examination for the course Foundations of Probability Theory. (MVE140 / MSA150)
Friday, 19 December 2008, 08.30-13.30 in an H-hall at Hörsalsvägen.
Examiner: Torgny Lindvall. Telephone connect. 3574 or mobile 0705-987486.
Teacher available at the examination site around 10.00 and 11.45.
Facilities: Dictionaries, from and into English.
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A completely solved problem gives 5 credit points.

We suppose that events and random variables are defined on a probability space that we call $(\Omega, \mathcal{F}, \mathbf{P})$.

1. Formulate and prove the Weak Law of Large Numbers for an IID sequence of random variables $X_1, X_2, \ldots$ with $Var[X_i] < \infty$.

2. We consider a simple random walk $W_1, W_2, \ldots$ starting at a: $W_0 = a$ and $W_n = a + \sum_1^n X_k$ for $n = 1, 2, \ldots$, where $X_1, X_2, \ldots$ are IID random variables taking values $+1$ and -1 only; $\mathbf{P}(X_k = 1) = p$. Prove, with $a = 1$, that if $p \leq \frac{1}{2}$ then $\mathbf{P}_1(H) = 1$ where $H = \{W_n = 0 \text{ for some } n \geq 1\}$ (The subscript "1" in $\mathbf{P}_1$ denotes that $W_0 = 1$).

3. Use MGF:s (moment generating functions) to prove that if X and Y are independent and normally distributed, then $X + Y$ is also normally distributed.

4. The random variable X has a finite second moment: $X \in L^2$. Prove that $Var[X] = 0$ only if there exists a constant c such that $\mathbf{P}(X = c) = 1$.

5. The IID random variables $X_1, X_2, \ldots$ are non-negative and have Laplace transform $L(\lambda)$ $(= \mathbf{E}[exp(-\lambda X_i)]) = exp(-\sqrt{\lambda})$ for $\lambda \geq 0$; such variables do exist! Let $S_n = \sum_1^n X_i, n = 1, 2, \ldots$. Show that there exists an $\alpha > 0$ such that the distributions of S_n / n^α are the same for all n.

6. The random variables $X_1, X_2, \ldots$ are IID: they are all $Exp(1)$-distributed, i.e., the common density function is $e^{-x}, x \geq 0$. Prove that $X_n / log(n) \to 0$ in probability as $n \to \infty$, but that $X_n / log(n)$ does not converge almost surely to 0 as $n \to \infty$.

Short solutions to Foundations of Probability Theory 19 Dec. 2008.
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1. Cf. Williams, Theorem 4.3.J, p.107.

2. Cf. Williams, 4.4.D, pp. 118-119. Let $x = \mathbf{P}_1(H)$. Conditioning on the first step of the random walk gives the equation $x = px^2 + q$ where $q = 1 - p$. But that equation has roots 1 and q/p, hence no root < 1.

3. Cf. Williams, 5.3 C and H-I, p. 147 and p. 152. If Z is $N(\mu_Z, \sigma_Z^2)$-distributed, then $M_Z(s) = \mathbf{E}[e^{sZ}] = exp(\mu_Z s + \frac{1}{2}(\sigma_Z s)^2)$. So if X and Y are normally distributed with parameters μ_X, σ_X^2 and μ_Y, σ_Y^2 respectively, then the "Independence means Multiply" rule implies: $M_{X+Y}(s) = exp(\mu_X s + \frac{1}{2}(\sigma_X s)^2) \cdot exp(\mu_Y s + \frac{1}{2}(\sigma_Y s)^2) = exp((\mu_X + \mu_Y)s + \frac{1}{2}((\sigma_X^2 + \sigma_Y^2)s)^2)$ so $X + Y$ is $N(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2)$-distributed. A reference to the uniqueness theorem concludes the solution.

4. Suppose that $Var[X] = 0$, and let μ denote $\mathbf{E}[X]$. We have $\mathbf{E}[(X - \mu)^2] = 0$. But a random variable $Y \geq 0$ which satisfies $\mathbf{E}[Y] = 0$ has to be 0 a.s. To see that, suppose $\mathbf{P}(Y > 0) > 0$. Then since $\mathbf{P}(Y > 1/n) \to \mathbf{P}(Y > 0)$ as $n \to \infty$, we can pick an n_0 so large that $\mathbf{P}(Y > 1/n_0) > 0$. But that means that $\mathbf{E}[Y] > 0$ since $Y \geq Y \cdot I(Y > 1/n_0) \geq (1/n_0) \cdot I(Y > 1/n_0)$, a contradiction. So: $1 = \mathbf{P}((X - \mu)^2) = 0) = \mathbf{P}(X = \mu)$.

5. We get that S_n has Laplace transform $L(\lambda)^n$ ("Independence means Multiply"). That implies: S_n/n^α has Laplace transform $exp(-\sqrt{\lambda/n^\alpha})^n = exp(-n\sqrt{\lambda/n^\alpha})$, which is independent of n if we let $\alpha = 2$. A reference to the uniqueness theorem for Laplace transforms completes the solution of the problem.

6. For an $Exp(1)$-distributed variable, Y say, we have that $\mathbf{P}(Y > y) = e^{-y}$ for all $y \geq 0$. To prove the convergence in probability, fix an $\epsilon > 0$. We get $\mathbf{P}(X_n/log(n) > \epsilon) = \mathbf{P}(X_n > \epsilon \cdot log(n)) = exp(-\epsilon \cdot log(n)) = n^{-\epsilon}$ which certainly converges to 0 as $n \to \infty$. But $\mathbf{P}(X_n/log(n) > \frac{1}{2}) = n^{-\frac{1}{2}}$ and $\sum_0^\infty n^{-\frac{1}{2}} = \infty$, so Borel-Cantelli's second lemma implies that $\mathbf{P}(X_n/log(n) > \frac{1}{2} \text{ i.o.}) = 1$, which means that we do not have a.s. convergence.