Examination for the course Foundations of Probability Theory.
Friday, 21 December 2007, 08.30-13.30 in the H house.
Examiner: Torgny Lindvall. Telephone connect. 3574 or mobile 0705-987486.
Teacher available at the examination site around 10.00 and 11.45.
Facilities: Dictionaries, from and into English.
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A completely solved problem gives 5 credit points.

We suppose that events and random variables are defined on a probability space that we call $(\Omega, \mathcal{F}, \mathbf{P})$.

1. The random variables $X$and Y are independent and have Poisson distributions. Use probability generating functions to find the distribution of $X + Y$.

2. We have an infinite sequence of events $A_1, A_2, \ldots$. Prove Boole's Inequality: $\mathbf{P}(\cup_1^\infty A_k) \le \sum_1^\infty \mathbf{P}(A_k)$.

3. Consider a symmetric Simple Random Walk $\mathbf{W} = (W_n)_0^\infty$ starting at an integer $a > 0$: $W_n = a + \sum_1^n X_k$, where the X_k's are the iid steps; $\mathrm{P}(X_k = 1) = \mathbf{P}(X_k = -1) = \frac{1}{2}$. What is the probability $\mathbf{P}(\mathbf{W}$ visits b before $0)$ where b is an integer $> a$? Hint: establish a difference equation; it is easily solved in the symmetric case!

4. The random variables N and $X_1, X_2, \ldots$ are independent and have finite expectations and variances, the X_k-variables are identically distributed and have expectation $= 0$, and N takes values in $\mathcal{Z}_+$. Let $S_n = \sum_1^n X_k$ for $n \in \mathcal{Z}_+$ $(S_0 = 0)$. Make a careful calculation of $Var[S_N]$ (every step commented!) in terms of the expectations and variances of N and X_k.

5. Give an example of two non-negative random variables $X$and Y with the same distribution such that $\mathbf{E}[X] < \infty$ but $\mathbf{E}[X \cdot Y] = \infty$.

6. Let $X_1, X_2, \ldots$ be iid with $\mathrm{E}[X_1] = 0$ and $\mathrm{E}[X_1^4] < \infty$. Formulate and prove the Strong Law of Large Numbers for $X_1, X_2, \ldots$.

FOR SHORT SOLUTIONS: PLEASE TURN PAGE!

Short solutions to Foundations of Probability Theory 21 Dec 2007.
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1. Cf. Williams [W], 5.2, pp. 143-144. Suppose X and Y have parameters λ and μ respectively. Due to the independence, $g_{X+Y}(s) = g_X(s) \cdot g_Y(s) = \exp(\lambda(s-1)) \cdot \exp(\mu(s-1)) = \exp((\lambda+\mu)(s-1))$ for $0 \le s \le 1$, which is the pgf of the Poisson distribution with parameter $\lambda + \mu$. Since a pgf determines the distribution on $\mathcal{Z}_+$, we have solved the problem.

2. Let $B_1 = A_1$, and for $k \ge 2$, let $B_k = A_k \setminus \cup_1^{k-1} A_i$. We have $\cup_1^\infty A_k = \cup_1^\infty B_k$. Since $B_k \subseteq A_k$ for all $k \ge 1$ and the B_k's are disjoint, we obtain $\mathbf{P}(\cup_1^\infty A_k) = \mathbf{P}(\cup_1^\infty B_k) = \sum_1^\infty \mathbf{P}(B_k) \le \sum \mathbf{P}(A_k)$.

3. Cf. [W], 4.4, pp. 117-118. With the notation used there, we have $x(k) = \frac{1}{2}(x(k-1)+x(k+1))$ for $1 \le k \le b-1$, which implies that $x(k+1)-x(k) = x(k)-x(k-1)$ for those values of k. It is now easy to see that $x(k) = k/b$ for all $0 \le k \le b$ since $x(0) = 0$ and $x(b) = 1$; hence the asked for probability equals a/b.

4. Cf. [W], 9.1, pp. 392-393. Use the arguments there to prove that $\mathbf{E}[S_N] = 0$. This means that $\mathbf{Var}[S_N] = \mathbf{E}[S_N^2]$. Using analogous arguments, we get $\mathbf{E}[S_N^2] = \sum_0^\infty \mathbf{E}[S_N^2|N=k] \cdot \mathbf{P}(N=k) = \sum_0^\infty \mathbf{E}[S_k^2] \cdot \mathbf{P}(N=k) = \sum_0^\infty \mathbf{Var}[S_k] \cdot \mathbf{P}(N=k) = \mathbf{Var}[X_1] \cdot \sum_0^\infty k \cdot \mathbf{P}(N=k) = \mathbf{Var}[X_1] \cdot \mathbf{E}[N]$.

5. We let $X = \exp((3/4) \cdot X')$, where X' is exponentially distributed with rate $\lambda = 1$. Then $\mathbf{E}[X] = \int_0^\infty \exp((3/4) \cdot x) \cdot \exp(-x) dx < \infty$, but if we let $Y = X$, then $\mathbf{E}[XY] = \mathbf{E}[X^2] = \infty$.
   Alternatively, let X have density $f(x) = C/(1+x^3)$ for $x \ge 0$ where C is a normalizing constant. Again: $\mathbf{E}[X] < \infty$ but $\mathbf{E}[X^2] = \infty$.

6. Cf. [W], 4.3, pp. 113-114.