Exam for the course Options and Mathematics (CTH[MVE095], GU[MMG810]) 2024/25

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August the 27th, 2025 (8.30-12.30)

REMARKS: (1) NO aids permitted (2) Write as clear as possible: if some step is not clearly readable it will be assumed to be wrong

Part I

1. Formulate backward recurrence formula (for binomial model) and prove that portfolio process is self-financing if and only if its value satisfies the formula. (3 p.)

2. Find and prove the probability density function for geometric Brownian motion.(3 p.)

3. Define and explain the definition of implied volatility of call option (Black-Scholes model). (3 p.)

4. You were appointed as a finance minister in a small but wealthy kingdom with the task of building a stock market. What can you do to make the market as frictionless as possible? Explain how the measures you take contribute to particular conditions of frictionlessness.

   Solution: This is an open end question, but one can suggest that one operate the stock market on the kingdoms funding, so that there will be no transaction costs. One can also have a reserve of stock and and take all bids at certain price to make transactions immediate and avoid bid/ask difference, but this would fix the price, so this measure would be rather negative for the market. What one can do instead is to make more people to trade, either by demanding that citizens would trade at least certain amount or by attracting foreign investors (for example by no taxes demanded). If one does all trading digital, guaranteed by state reserves, one can introduce trading in fractions. (3 p.)

Part II

1. Let $S(t) > 0$ be the price at time t of a non-dividend paying stock.

$$f(S) = \begin{cases} 0, & \text{for } x < 100; \\ x - 100, & \text{for } 100 < x < 200; \\ 200 - \frac{1}{2}x, & \text{for } 200 \leq x \leq 400; \\ 0, & \text{for } x \geq 400. \end{cases}$$

   (a) Find a constant portfolio $\mathcal{A}$ of European calls on a stock and, if applicable, the stock itself and cash, which at time T has the value $V(\mathcal{A}) = f(S(T))$. (3 p.)

   Solution: 1 long call with strike 100, 1.5 short calls with strike 200, 0.5 long calls with strike 400

   (b) Find a constant portfolio $\mathcal{A}$ of European puts on a stock and, if applicable, the stock itself and cash, which at time T has the value $V(\mathcal{A}) = f(S(T))$. (2 p.)

   Solution: 1 long put with strike 100, 1.5 hort put with strike 200, 0.5 long put with strike 400

   (c) If annual rate of return on zero coupon 2 years maturity bond, is 10%. What is the face of the bond, if the current price of the bond is 1000kr. (1 p.)

   Solution: $1000 \cdot 1.1^2 = 1210$

2. Given is a stock of current value 400kr, we use to price it a binomial model in which at every time interval the stock has log-return $\log \frac{3}{2}$ with probability 1/3 and log-return $\log \frac{1}{2}$ with probability 2/3. We assume that a risk-free zero coupon bond used in the model has log-return $\log \frac{5}{4}$.

(a) Find the price at time $t = 0$ of a European call with strike 200, which matures in 2 time intervals. (2 p.)

Solution: $q = \frac{5/4-1/2}{3/2-1/2} = 3/4$, $S(0) = 400$, $S(1,u) = 200$, $S(1,d) = 600$, $S(2,u,u) = 100$, $S(2,u,d) = S(2,d,u) = 300$, $S(2,d,d) = 900$. $\Pi(2,u,u) = 0$, $\Pi(2,u,d) = \Pi(2,d,u) = 100$, $\Pi(2,d,d) = 700$, $\Pi(1,d) = 4/5(3/4 \cdot 700 + 1/4 \cdot 100) = 2200/5 = 440$, $\Pi(1,u) = 4/5(3/4 \cdot 100 + 0) = 60$, $\Pi(0) = 4/5(3/4 \cdot 440 + 1/4 \cdot 60) = 1920/5 = 384$. Answer: the price is 384.

(b) Find the price at time $t = 0$ of a American put with strike 300, which matures in 2 time intervals. (2 p.)

Solution: See part of computation above. $\Pi(2,u,u) = 200$ $\Pi(2,u,d) = \Pi(2,d,u) = \Pi(2,d,d) = 0$**,** $\Pi(1,d) = max(0,0) = 0$**,** $\Pi(1,u) = max(100, 4/5(3/4 \cdot 0 + 1/4 \cdot 200)) = 100$**,** $\Pi(0) = max(0, 4/5(3/4 \cdot 0 + 1/4 \cdot 100)) = 20$**. Answer: The price is** 20**.**

(c) Find the price at time $t = 0$ of an Asian put which matures in 2 time intervals and can be executed only at time 2 with the pay-off $Y = (300 - \frac{S(1)+S(2)}{2})_+$. (2 p.)

Solution: $\Pi(2,d,d) = \Pi(2,d,u) = 0$**,** $\Pi(2,u,d) = 50$**,** $\Pi(2,u,u) = 150$**,** $\Pi(1,d) = 0$**,** $\Pi(1,u) = \frac{4}{5}(\frac{3}{4}50 + \frac{1}{4}150) = 60$**,** $\Pi(0) = \frac{4}{5}(\frac{3}{4}0 + \frac{1}{4}60) = 12$**.**

3. Given a stock of current value 1500kr and constant log-volatility $\sigma = 3$, which is supposed to pay 8% of is market value as dividends in 2 years time (at $t_d = 2$) use Black-Scholes model to set prices on following derivatives based on the stock. The interest rate on the risk-free asset is 0.2. (Observe that you are allowed to give an answer as an expression combining elementary functions, Φ-function and numbers without evaluating it.)

   (a) A European call with strike 1000kr and maturity in 1 year. (2 p.)
   Solution: $1500\Phi(d_+) - 1000e^{-0.2}\Phi(d_-)$, where $d_\pm = (\log(1.5) + (0.2 \pm 4.5))/3$.

   (b) A standard Europian derivative with pay-off $Y = (S(T))^5$ and maturity in 1 years. (2 p.)
   Solution: $\Pi = e^{-0.2} \int (1500 e^{(0.2+4,5)+3y)})^5 e^{-y^2/2} \frac{dy}{\sqrt{2\pi}} = e^{-0.5+1+22.5+15^2} (1500)^5 \int e^{-(y-1}$
   $e^{23+15^2}(1500)^5$.

(c) A European put with strike 1300kr and maturity in 4 years.(2 p.)
Solution: $0.08 \cdot 1500 = 120$, $1500 - 120 = 1380$. $1300e^{-0.2}\Phi(-d_-) - 1380\Phi(-d_+)$, where $d_\pm = (\log(\frac{138}{130}) + (0.2 \pm 4.5)4)/6$.