Exam for the course Options and Mathematics (CTH[MVE095], GU[MMG810]) 2024/25

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January the 18th, 2025 (8.30-12.30)

REMARKS: (1) NO aids permitted (2) Write as clear as possible: if some step is not clearly readable, it will be assumed to be wrong

Part I

1. Formulate and prove the backward recurrence formula for the self-financing portfolio process in a binomial model. (3 p.)
2. Find and prove the probability density function for geometric Brownian motion.(3 p.)
3. Define and explain the use of geometric Brownian motion in the course. (3 p.)
4. Consider two stocks: Stock A is a Finish green technology start-up; Stock B is an established French car manufacturer. Which stock trading on the US stock market will be more close to being arbitrage free? What aspects can affect the answer? (Observe that your argument rather than the answer will be judged.) (3 p.)

Part II

1. Let $S(t) > 0$ be the price at time t of a non dividend paying stock.

$$f(S) = \begin{cases} 0, & \text{for } x < 100; \\ x - 100, & \text{for } 100 \leq x \leq 200; \\ 100, & \text{for } x > 200; \end{cases}$$

   (a) Find a constant portfolio $\mathcal{A}$ of European calls on a stock and, if applicable, the stock itself and cash, which at time T has the value $V(\mathcal{A}) = f(S(T))$. (3 p.)

   Suggested answer: One call with strike 100 in long position, and one call with strike 200 in short position.

   (b) Find a constant portfolio $\mathcal{A}$ of European puts on a stock and, if applicable, the stock itself and cash, which at time T has the value $V(\mathcal{A}) = f(S(T))$. (2 p.)

   Suggested answer: 100 units of cash/bond with face value 100 in long position, one put with strike 100 in long position and one put with strike 200 in short position.

   (c) If annual rate of return on zero coupon bond with maturity in 3 years issued now, is 10%. If the face of the bond is 1331kr, what is the current price of the bond? (1 p.)

   Suggested answer: If the current price of the bond is X then $X(1 + 0.1)^3 = 1331$. As $1.1^3 = 1.331$, $X = 1000$kr.

2. Given is a stock of current value 900kr, we use a binomial model to price it: In the model at every time interval the stock has log-return $\log \frac{5}{3}$ with probability 1/3 and log-return $\log \frac{2}{3}$ with probability 2/3. We assume that a risk-free zero coupon bond used in the model has log-return $\log \frac{4}{3}$.

   (a) Find the price at time $t = 0$ of a European put with strike 800, which matures in 2 time intervals. (2 p.)

   Suggested solution: $e^u = 5/3$, $e^d = 2/3$, $e^r = 4/3$. $q_u = 2/3, q_d = 1/3$. $S(0) = 900, S(1, U) = 1500, S(1, d) = 600, S(2, u, u) = 2500, S(2, u, d) = S(2, d, u) = 1000, S(2, d, d) = 400$. $\Pi(2, u, u) = 0, \Pi(2, u, d) = \Pi(2, d, u) = 0, \Pi(2, d, d) = 400$. $\Pi(1, u) = 3/4 \cdot (2/3 \cdot 0 + /3 \cdot 0) = 0$, $\Pi(1, d) = 3/4 \cdot (2/3 \cdot 0 + 1/3 \cdot 400) = 100$.

$\Pi(0) = 3/4 \cdot (2/3 \cdot 0 + 1/3 \cdot 100) = 25$. Answer: $P(0, 900, 2, 800) = 25$.

(b) Find the price at time $t = 0$ of an American put with strike 1000, which matures in 2 time intervals. (2 p.)

Suggested solution: Some calculations have already been done in a). $Y(0) = 100, Y(1,u) = 0, Y(1,d) = 400, Y(2,u,u) = 0, Y(2,u,d) = Y(2,d,u) = 0, Y(2,d,d) = 600$. $\Pi(2,u,u,) = 0, \Pi(2,u,d) = \pi(2,d,u) = 0, \Pi(2,d,d) = 600$. $\Pi(1,u) = max(0,0) = 0, \Pi(1,d) = max(400, 3/4(2/3 \cdot 0 + 1/3 \cdot 600)) = 400$, $\Pi(0) = max(100, 3/4(2/3 \cdot 0 + 1/3400)) = 100$.

(c) Find the price at time $t = 0$ of an Asian call which matures in 2 time intervals and can be executed only at time 2 with the pay-off $Y = (\frac{S(1)+S(2)}{2} - 1000)_+$. (2 p.)

Suggested solution: Some calculations have already been done in a). $\Pi(2,u,u) = 1000, \Pi(2,u,d) = 250, \Pi(2,d,u) = 0, \Pi(2,d,d) = 0$.So, $\Pi(1,u) = 3/4 \cdot (2/3 \cdot 1000 + 1/3 \cdot 250) = 2250/4, \Pi(1,d) = 0$. $\Pi(0) = 3/4 \cdot (2/3 \cdot 2250/4 + 1/3 \cdot 0) = 2250/8$.

3. Given a stock of current value 1000kr and constant log-volatility $\sigma = 3$, which is supposed to pay 4% of is market value as dividends in 3 years time (at $t_d = 3$) use Black-Scholes model to set prices on following derivatives based on the stock. The interest rate on the risk-free asset is $r = 0.02$. (Observe that you are allowed to give an answer as an expression combining elementary functions, Φ-function and numbers without evaluating it. If the answer is an integral, you should evaluate the integral in order to obtain full points.)

(a) A European call with strike 800kr and maturity in 1 year. (2 p.)

Suggested solution: $1000\Phi(d_/(+)) - 800e^{-0.02}$, where $d_\pm = \frac{\log(\frac{1000}{800})+(0.02\pm 9/2)}{3}$.

(b) A standard European derivative with pay-off $Y = (S(T))^3$ and maturity in 2 years. (2 p.)

Suggested solution: The price can be evaluated by the formula $e^{-0.04}\int_{\mathbf{R}}(1000e^{(0.02-\frac{1}{2}9)2+3\sqrt{2}y})^3 e^{-\frac{1}{2}y^2}dy/\sqrt{2\pi} = 10^9 e^{-9}\int_{\mathbf{R}} e^{9\sqrt{2}y-\frac{1}{2}y^2}dy/\sqrt{2\pi} = 10^9 e^{-9+162} = 10^9 e^{153}$

(c) A European put with strike 1300kr and maturity in 5 years.(2 p.)

Suggested solution: $960\Phi(d_/(+)) - 1300e^{-0.1}$, where $d_\pm = \frac{\log(\frac{960}{1300})+5(0.02\pm 9/2)}{3\sqrt{5}}$.