Exam for the course Options and Mathematics (CTH[MVE095], GU[MMG810]) 2022/23

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August the 23d, 2023 (8.30-12.30)

REMARKS: (1) NO aids permitted (2) Write as clear as possible: if some step is not clearly readable it will be assumed to be wrong

Part I

1. Show that in binomial 1-period model a self-financing arbitrage portfolio exists if and only if $r \notin (d, u)$ (Theorem 2.4). (3 p.) **(see the book)**

2. State and derive formula for Black-Scholes pricing of an European option with a dividend before the time of maturity (Theorem 6.18).(3 p.) **(see the book) Due to a typo in initial task also Theorem 6.13 and Theorem 6.1 are acceptable.**

3. Define and explain binomial risk-neutral price of American derivatives as maximum of expectation at time $t = 0$ (Definition 5.19). (3 p.) **(see the book)**

4. Name at least two reasons to buy options on a stock?

   Proposed solution: One can secure ones investment in stock by buying an option, or one can secure holding a short position in a stock. On the other hand, one can benefit by guessing moving of a stock price without investing in buying the stock (one take though greater risk if the stock price doesn't move as one expected).

(3 p.)

Part II

1. Let $S(t) > 0$ be the price at time t of a non-dividend paying stock.

$$f(S) = \begin{cases} 100, & \text{for } x < 100; \\ 200 - x, & \text{for } 100 \leq x \leq 200; \\ 0, & \text{for } x > 200; \end{cases}$$

   (a) Find a constant portfolio $\mathcal{A}$ of European calls on a stock and, if applicable, the stock itself and cash, which at time T has the value $V(\mathcal{A}) = f(S(T))$. (3 p.)

   Proposed solution: 100 units in cash, short 100 units in calls with strike 100, and long in 100 call units with strike 200.

   (b) Find a constant portfolio $\mathcal{A}$ of European puts on a stock and, if applicable, the stock itself and cash, which at time T has the value $V(\mathcal{A}) = f(S(T))$. (2 p.)

   Proposed solution: 100 units short in puts with strike 100 and 100 units long in puts with strike 200.

   (c) If annual rate of return on zero coupon 2 years maturity bond, is 10%. The face of the bond is 12100kr, what is the current price of the bond. (1 p.) **Proposed solution:** $x \cdot (1.1)^2 = 12100$, $1,21x = 12100$, $x = 10000$. So, current price is 10000kr.

2. Given is a stock of current value 900kr, we use to price it a binomial model in which at every time interval the stock has log-return $\log \frac{4}{3}$ with probability 1/3 and log-return $\log \frac{2}{3}$ with probability 2/3. We assume that a risk-free zero coupon bond used in the model has a constant price

   (a) Find the price at time $t = 0$ of a European call with strike 600, which matures in 2 time intervals. (2 p.)

   Proposed solution: $q_u = \frac{1-2/3}{4/3-2/3} = 1/2$. $S(0) = 900, S(1,u) = 1200.S(1,d) = 600, S(2,u,u) = 1600, S(2,u,d) = S(2,d,u) = 800, S(2,d,d) = 400$. $Y(2,u,u) = 1000, Y(2,u,d) = Y(2,d,u) = 200$. $\Pi(0) = (1/41000 + 21/4200) = 350$.

(b) Find the price at time $t = 0$ of a American put with strike 700, which matures in 2 time intervals. (2 p.)

Proposed solution: As American options can be executed at any time we should consider yield at all times. We see that yield is zero except for $Y(1, d) = 100$ and $Y(2, d, d) = 300$. At time 1 $\Pi(1, u) = 0, \Pi(1, d) = max(Y(1, d), 1/2\Pi(2, d, u)+1/2\Pi(2, d, d)) = max(100, 150) = 150$. $\Pi(0) = 150/2 = 75$.

(c) Find the price at time $t = 0$ of an Asian put which matures in 2 time intervals and can be executed only at time 2 with the pay-off $Y = (900 - \frac{S(1)+S(2)}{2})_+$. (2 p.)

Proposed solution: $Y(2, u, u) = 0, Y(2, u, d) = 0, Y(2, d, u) = 200, Y(2, d, d) = 400$, so $\Pi(0) = (1/4200 + 1/4400 = 150$.

3. Given a stock of current value 1000kr and constant log-volatility $\sigma = 4$, which is supposed to pay 3% of is market value as dividends in 3 years time (at $t_d = 3$) use Black-Scholes model to set prices on following derivatives based on the stock. The interest rate on the risk-free asset is 0.1. (Observe that you are allowed to give an answer as an expression combining elementary functions, Φ-function and numbers without evaluating it.)

(a) A European call with strike 800kr and maturity in 1 year. (2 p.)

Proposed solution: $d_+ = (\log(1000/800) + (0.1 + 16/2))/4$, $d_- = (\log(1000/800) + (0.1 - 16/2))/4$. Price is $1000\Phi(d_+) - 800e^{-0.1}\Phi(d_-)$.

(b) A standard European derivative with pay-off $Y = (S(T))^3$ and maturity in 2 years. (2 p.)

Proposed solution:

$V(0) = e^{-0.2} \int_{\mathbf{R}}(1000e^{(0.1-4^2/2)2+4\sqrt{2}y})^3 e^{-y^2/2}dy/\sqrt{2\pi} =$

$10^9 e^{-0.2+7.9\cdot 6} \int_{\mathbf{R}} e^{12\sqrt{2}y-y^2/2}dy/\sqrt{2\pi} =$

$10^9 e^{-0.2+7.9\cdot 6+288} \int_{\mathbf{R}} e^{-(y-12\sqrt{2})^2/2}dy/\sqrt{2\pi} = 10^9 e^{-0.2+7.9\cdot 6+288}$.

(c) A European put with strike 1300kr and maturity in 5 years.(2 p.)

Proposed solution: To account for dividends we replace 1000 för 970 $d_+ = (\log(970/800)+(0.1+16/2))/4$, $d_- = (\log(970/800)+(0.1 - 16/2))/4$.

$P(0, 1000, 1300, 5) = \Phi(-d_-)1300e^{-0.5} - \Phi(d_+)970$.