# Exam for the course Options and Mathematics (CTH[MVE095], GU[MMG810]) 2022/23

For questions call the examiner at +46 (0)31 772 35 70

April the 4th, 2023 (8.30-12.30)

REMARKS: (1) NO aids permitted (2) Write as clear as possible: if some step is not clearly readable it will be assumed to be wrong

## Part I

1. Formulate and prove put-call parity for a non-divident paying stock (Theorem 1.2). (3 p.)

   (see the textbook)

2. Describe a self-financing replicating portfolio process for European derivative with pay-off $Y$ at the time of maturity $T = N$ (Theorem 3.3). (3 p.)

   (see the textbook)

3. Define geometric Brownian motion and how it is connected to this course(Definition 6.7). (3 p.)

   (see the textbook plus notice that stock price is supposed to change as a geometric Brownian motion (or, which is equivalent that log-return is a Brownian motion).

4. Assume we have two stocks. The stock A is of a national enterprise working with staffing of retiring homes. The stock B is an international enterprise developing AI-solution for individualized medication, which

participated in development of vaccine for COVID-19. Which stock, in your opinion, is better described by the theory of this course? Explain your answer.

**Lösningsförslag:** Stock B should be more exposed for traders due to wider exposure (international vs. national) and being due to its activity and sector more visible on the news. Thus, stock B is more traded, and is in conditions more close to both friction-free and arbitrage-free market. (3 p.)

# Part II

1. Let $S(t) > 0$ be the price at time t of a non-dividend paying stock.

$$f(S) = \begin{cases} 200 - x, & \text{for } x < 100; \\ x, & \text{for } 100 \leq x \leq 300; \\ 300, & \text{for } x > 300; \end{cases}$$

   (a) Find a constant portfolio $\mathcal{A}$ of European calls on a stock and, eventually, the stock itself and cash, which at time $T$ has the value $V(\mathcal{A}) = f(S(T))$. (3 p.)

   **Suggested solution:** Pay-off can be written as $300 - P(300, T) + 2P(100, T)$. As we want to have only calls, we can use put-call parity: $C(K, T) - P(K, T) = S(T) - K$. Thus, $300 - P(300, T) + 2C(100, T) = 300 + S(T) - 300 - C(300, T) + 2(-S(T) + 100 + C(100, T)) = 200 - S(T) + 2C(100, T) - C(300, T)$.

   (b) Find a constant portfolio $\mathcal{A}$ of European puts on a stock and, evenually, the stock itself and cash, which at time $T$ has the value $V(\mathcal{A}) = f(S(T))$. (2 p.)

   **Suggested solution:** Pay-off can be written as $300 - P(300, T) + 2P(100, T)$.

   (c) If annual rate of return on zero coupon bond and 2 years maturity, issued today, is 10%. The current price of the bond is 1000kr, what is the face value of the bond. (1 p.)

   **Suggested solution:** The price of the bond at the time of maturity is $(1 + 0, 1)^2 1000 = 1, 21 \cdot 1000 = 1210$, and this should be its face value.

(Note: Due to a typo in this question during the exam, the grade bounds were recalculated to exclude this question.)

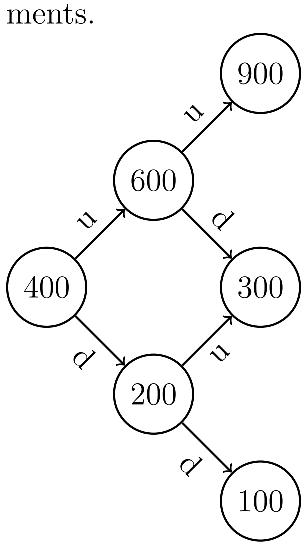
2. Given is a stock of current value 400kr, we use to price it a binomial model in which at every time interval the stock has log-return $\log \frac{3}{2}$ with probability 1/2 and log-return $\log \frac{1}{2}$ with probability 1/2. We assume that a risk-free zero coupon bond used in the model has at every time interval log-return $\log \frac{5}{4}$.

   (a) Find the price at time $t = 0$ of a European call with strike 500, which matures in 2 time intervals. (2 p.)

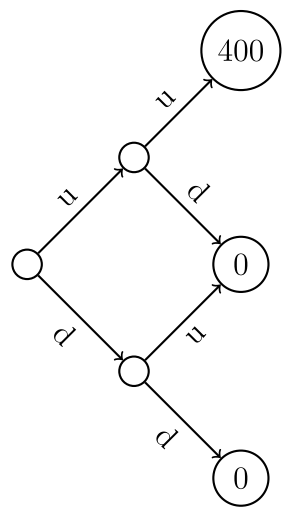
   **Suggested solution:**

   We start from finding risk-neutral probabilities: $e^u = 3/2$, $e^d = 1/2$, $q_u = \frac{e^{\log(5/4)} - e^{\log(1/2)}}{e^{\log(3/2)} - e^{\log(1/2)}} = \frac{3}{4}$, $q_d = 1 - q_u = 1/4$

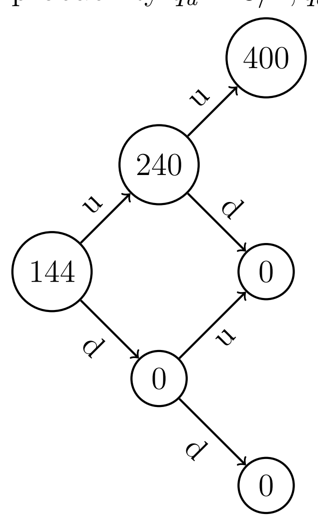
   We draw a (factorized) graph of stock prices for different developments.




   We draw a graph of pay-off for European call with strike 500:




We count backwards the price of the call by using risk-neutral probability $q_u = 3/4, q_d = 1/4$, $e^r = 4/5$.

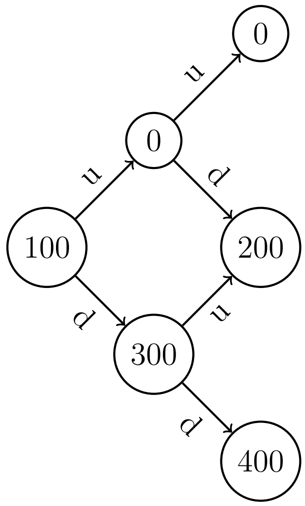



So, the price of the call is 144kr.

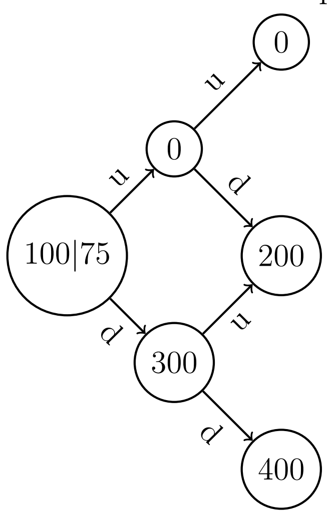
(b) Find the price at time $t = 0$ of a American put with strike 500, which matures in 2 time intervals. (2 p.)

**Suggested solution:**

We draw a graph of pay-off for American put with strike 500, notice that we assume we can execute at any time (in some interpretations we can not execute at time 0):




We count backwards the price of the put by using risk-neutral probability $q_u = 3/4, q_d = 1/4$, $e^r = 4/5$, and taking the maximum between the calculated price for held option and price of execution.

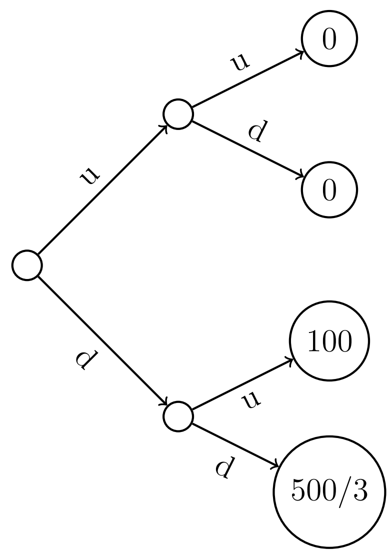



So, the price of the put is 100kr eller 75kr.

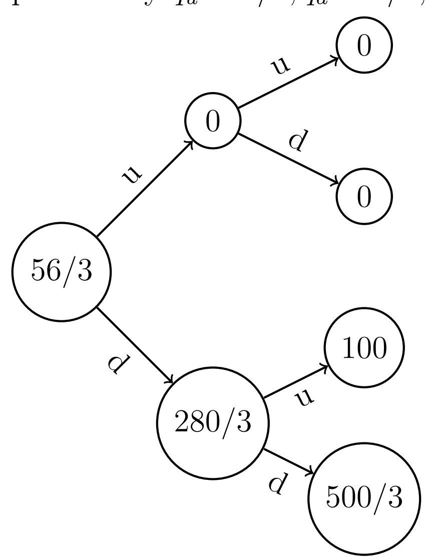
(c) Find the price at time $t = 0$ of an Asian put which matures in 2 time intervals and can be executed only at time 2 with the pay-off $Y = (400 - \frac{S(0)+S(1)+S(2)}{3})_+$. (Keep fractions in your calculations.) (2 p.)

**Suggested solution:**

We draw a graph of pay-off for the Asian put :




We count backwards the price of the put by using risk-neutral probability $q_u = 3/4, q_d = 1/4$, $e^r = 4/5$.




So, the price of the put is 56/3kr.

3. Given a stock of current value 1000kr and constant log-volatility $\sigma = 9$, which is supposed to pay 3% of is market value as dividends in 3 years time (at $t_d = 3$) use Black-Scholes model to set prices on following derivatives based on the stock. The interest rate on the risk-free asset is 0.01. (Observe that you are allowed to give an answer as an expres-

sion combining elementary functions, $\Phi$-function and numbers without evaluating it.)

(a) A European call with strike 900kr and maturity in 1 year. (2 p.)

**Suggested solution:** The European call has price $1000\Phi(d_{(+)} - 900e^{-0.01}\Phi(d_{(-)}$, where $d_{(+)} = \frac{\log(1000/900)+0.01+81/2}{9}$ and $d_{(-)} = \frac{\log(1000/900)+0.01-81/2}{9}$

(b) A standard Europian derivative with pay-off $Y = (S(T))^5$ and maturity in 2 years. (2 p.)

**Suggested solution:** The price is $e^{-0.02}\int(1000e^{(0.01-81/2)2+9\sqrt{2}y})^5 e^{-1/2y^2}dy/\sqrt{2\pi}$
$= 10^{15}e^{-0,02+10(0.01-81/2)}\int e^{45\sqrt{2}y-1/2y^2}dy\sqrt{2\pi} = 10^{15}e^{-0,02+10(0.01-81/2)+(45)^2}\int e^{-\frac{1}{2}(y-45\sqrt{(2}}$
$10^{15}e^{-0,02+10(0.01-81/2)+(45)^2}$,

(c) A European put with strike 1200kr and maturity in 5 years.(2 p.)

Suggested solution: Due to once paid dividend the price of the stock shold be adjusted to $1000 - 0.03 \cdot 1000 = 970$, so the price of the option is $1200e^{-0.05}\Phi(-d_{(-)}) - 970\Phi(-d_{(+)})$, where $d_{(-)} = \frac{970/1200+5(0.01-81/2)}{9\sqrt{5}}$ and $d_{(+)} = \frac{970/1200+5(0.01+81/2)}{9\sqrt{5}}$.