Exam for the course
Options and Mathematics
(CTH[MVE095], GU[MMG810]) 2022/23

For questions call the examiner at +46 (0)31 772 35 70

January 12th, 2023 (8.30-12.30)

REMARKS: (1) NO aids permitted (2) Write as clear as possible: if some step is not clearly readable it will be assumed to be wrong

Part I

1. In Black-Scholes model price of a stock $S(t)$ is given by a geometric Brownian motion. Find the density of the random variable $S(t)$ (Theorem 6.10). (3 p.)

2. What is a Black-Scholes price of a standard European derivative with pay-off $Y = g(S(T))$, assuming that underlying stock pays dividend before the time of maturity (Theorem 6.18). (3 p.)

3. What is an American put option? Define optimal execution time for an American put option (Definition 1.2). (3 p.)

   Comment: You should both give the definition in terms of formulas, and an explanation that at the optimal execution time the price of derivative is equal to the pay-off for its execution, so we execute it without a loss.

4. Assume we have two markets with equal amount equally active traders. At market A only one stock is traded. On market B 1000 stocks are traded. Which market will be more close to being arbitrage free? Explain your answer. (3 p.)

A possible argument: As in the first market more trading happens in the same stock arbitrage opportunity are more likely to be explored and removed from the market. Thus, market A is more close to be arbitrage free.

Part II

1. Let $S(t) > 0$ be the price at time t of a non-dividend paying stock.

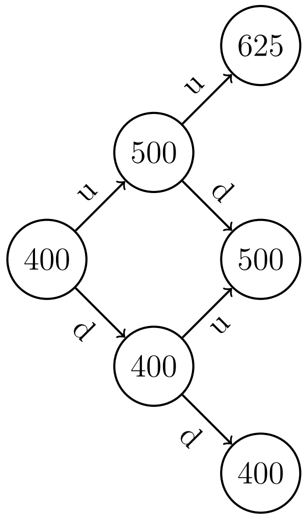
   (a) Find a constant portfolio $\mathcal{A}$ of European calls on a stock, which at time T has the value $V(\mathcal{A}) = (100 - |200 - S(T)|)_+$. (3 p.)
   Solution: Portfolio should be long in one call on the stock with strike 100, short in two calls on the stock with strike 200, and long in one call on the stock with strike 300. All calls should have maturity at time T.

   (b) Find a constant portfolio $\mathcal{A}$ of European puts on a stock, which at time T has the value $V(\mathcal{A}) = (100 - |200 - S(T)|)_+$. (2 p.)
   Solution: Portfolio should be long in one put on the stock with strike 100, short in two puts on the stock with strike 200, and long in one put on the stock with strike 300. All puts should mature at time T.

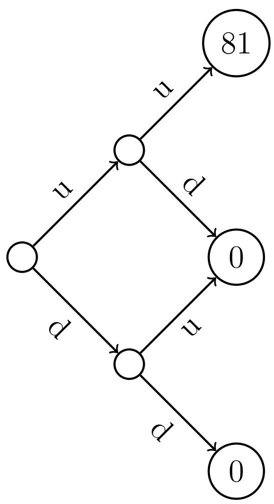
   (c) If annualized rate of return on zero coupon bond with face value 1500kr and 10 years maturity, issued today, is 5%, what is the price of the bond now (at the moment of issue). (1 p.)
   Solution: We see that $0.05 = \frac{V(10)-V(0)}{10V(0)}$. We also know that $V(10) = 1500$. Thus, $0,5V(0) = 1500 - V(0)$, i.e. $1,5V(0) = 1500$. Answer: $V(0) = 1000$kr.

2. Given is a stock of current value 400kr, we use to price it a binomial model in which at every time interval the stock has log-return $\log \frac{5}{4}$ with probability 2/3 and log-return 0 with probability 1/3. We assume that a risk-free zero coupon bond used in the model has at every time interval log-return $\log \frac{9}{8}$.

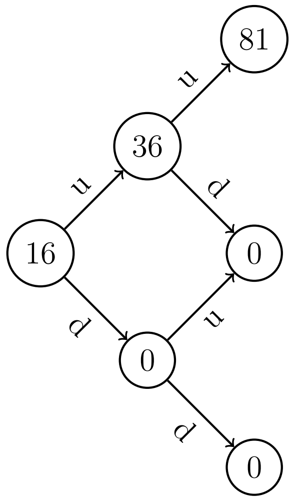
   (a) Find the price at time $t = 0$ of a European call with strike 544, which matures in 2 time intervals. (2 p.) Solution: We draw a (factorized) graph of stock prices for different developments.



We draw a graph of pay-off for European call with strike 544:



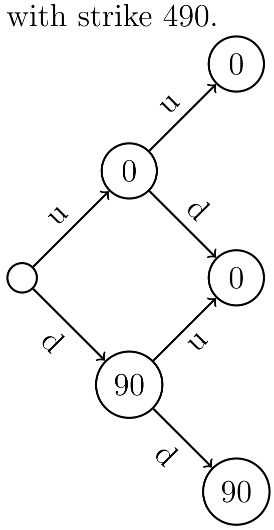
We count backwards the price of the call by using risk-neutral probability $q_u = \frac{9/8-1}{5/4-1} = \frac{1/8}{1/4} = 1/2$.



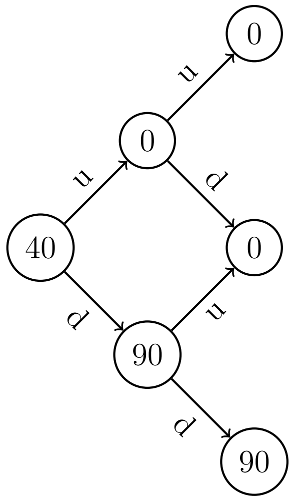
So, the price of the call is 16kr.

(b) Find the price at time $t = 0$ of a American put with strike 440, which matures in 2 time intervals. (2 p.)

Solution: We can use the graph for prices from previous subquestion. We draw the graph of pay-offs for the American put with strike 490.

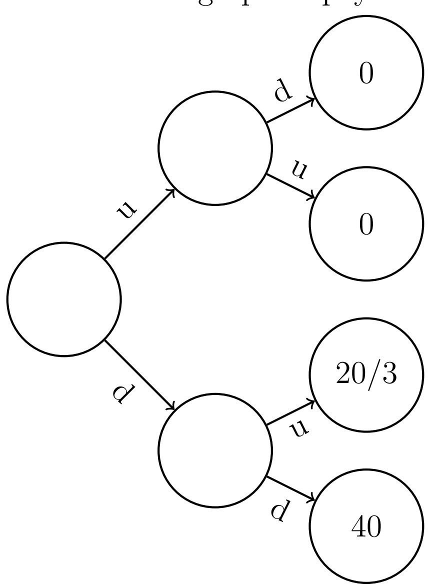


We find graph for the American put prices counting backward (observe that in the dawn node at time 1 we have an optimal execution time):

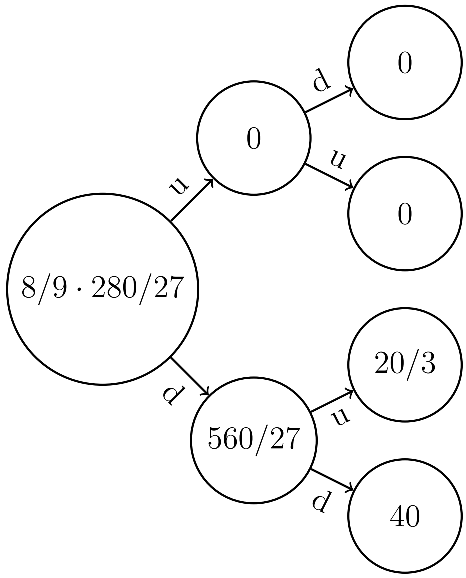


(c) Find the price at time $t = 0$ of an Asian option which matures in 2 time intervals, can be executed only at time 2 and has pay-off $Y = (440 - \frac{S(0)+S(1)+S(2)}{3})_+$. (2 p.)

Solution: We use the graph for the price of the stock from earlier and draw the graph of pay-off:



So, we can derive the graph of prices:



So, we see that the price of the derivative is $8/9 \cdot 280/27$.

3. Given a stock of current value 1000kr and log-volatility $\sigma = 4$, which is supposed to pay once 5% of is market value as dividends in 2 years time (at $t_d = 2$) use Black-Scholes model to set prices on following derivatives based on the stock. The interest rate on the risk-free asset is 0.1.

   (a) A European call with strike 950kr and maturity in 1 year. (2 p.)
   Solution: The European call has price of $1000\Phi(d_{(+)}) - 950e^{0.1}\Phi(d_{(-)})$, where $d_{(+)} = (\log(1000/950)+0.1+8)/4$ and $d_{(-)} = (\log(1000/950)+0.1-8)/4$.

   (b) A standard Europian derivative with pay-off $Y = (S(t))^3$ and maturity in 1 year. (2 p.)
   Solution: $\Pi(0) = e^{-0.1}\int_{\mathbb{R}}(1000e^{(0.1-8)+4y})^3 e^{-1/2y^2}dy/\sqrt{2\pi}$
   $= 10^9 e^{0.2-24}\int_{\mathbb{R}} e^{-1/2(y^2-24y)}dy/\sqrt{2\pi} = 10^9 e^{0.2-24+72}\int_{\mathbb{R}} e^{-1/2(y^2-24y+144)}dy/\sqrt{2\pi} = 10^9 e^{48.2}$.

   (c) A European put with strike 1100kr and maturity in 3 years.(2 p.)
   Solution: A European put with strike 1100 and matures in 3 years on the stock which once pays dividends of 5% once has price of $\Phi(-d_{(-)})1100e^{-0.3} - \Phi(-d_{(+)})950$, where $d_{(+)} = (\log(950/1100) + 3(0.1+8))/4\sqrt{3}$ and $d_{(+)} = (\log(950/1100) + 3(0.1-8))/4\sqrt{3}$.