Exam for the course "Options and Mathematics" (CTH[*MVE095*], GU[*MMG810*]) 2021/22

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August 23$^{\mathrm{rd}}$, 2022 (8.30-12.30)

REMARKS: (1) NO aids permitted (2) Write as clear as possible: if some step is not clearly readable it will be assumed to be wrong.

Part I

1. Prove the put-call parity on arbitrage-free markets (max 3 points). Prove that any European derivative in an arbitrage-free binomial market can be replicated by a self-financing predictable portfolio process (max 3 points).

2. Give and explain the definition of optimal exercise curve of the American put option (max 3 points).

3. Decide whether the following statements are true or false in an arbitrage free market and explain your answer (max 3 points):

   (a) The return on an American put exercised at an optimal exercise time is always positive.

   (b) The return on an American put exercised at a time that is not optimal is always negative.

   (c) If an American put is exercised at the first optimal exercise time, the return on the replicating portfolio of the seller is zero.

   Solution: (a) False. For instance, at maturity it is optimal to exercise when $S(T) < K$, but the return is negative if $K - S(T) - \Pi_0 < 0$, i.e., $S(T) > K - \Pi_0$, where Π_0 is the premium payed by the buyer of the put. By continuity there are also optimal exercise times prior, but close, to maturity at which the return is negative. (b) False. The return on the American put exercised at time t is $(K - S(t))_+ - \Pi_0$, which is positive for $S(t) < K - \Pi_0$, regardless of weather t is an optimal exercise time or not. (c) True. If the American put is exercised at the first optimal exercise time, the seller cannot withdraw any cash from the replicating portfolio and thus the return for the seller is zero.

Part II

1. An investor wants to set-up a constant portfolio on European options whose payoff $V(T)$ at time T in terms of the underlying stock price is given as in the figure on the next page. Find an example of such portfolio (max 3 points). Assuming that the stock price follows a geometric Brownian motion with volatility σ and mean of log-returns α, compute the range

of α such that the portfolio payoff at time T is non-negative (i.e., $V(T) \geq 0$) with probability larger than $1/2$ (max 3 points).

Solution: An example of portfolio is $(C(5), -2C(10), C(20))$, where $C(K)$ denotes the call with strike K and maturity T. The payoff of the portfolio at time T is non-negative if $S(T) < 15$, and we have

$$\mathbb{P}(S(T) < 15) = \mathbb{P}(S_0 e^{\alpha T + \sigma W(T)} \leq 15) = \mathbb{P}(G \leq \frac{\log \frac{15}{S_0} - \alpha T}{\sigma \sqrt{T}}) = \Phi(\frac{\log \frac{15}{S_0} - \alpha T}{\sigma \sqrt{T}})$$

where $G = W(T)/\sqrt{T} \in \mathcal{N}(0,1)$ and Φ is the standard normal distribution. Since $\Phi(z) > 1/2$ if and only if $z > 0$, the answer to the second question is

$$\alpha < \frac{1}{T} \log \frac{15}{S_0}.$$

2. Let $N = 2$ and $e^u = 2$, $e^d = 1/2$, $e^r = 1$, $S_0 = 8$, $p \in (0,1)$. Consider a European derivative with maturity $T = 2$ and payoff

$$Y = \left(\frac{1}{3} \Big(\sum_{i=0}^{2} S(i) \Big) - \frac{29}{3} \right)_+ ,$$

which is an example of Asian call option. Compute the price of the derivative at times $t = 0, 1, 2$ and the position on the stock in the hedging portfolio in the time intervals $[0,1]$, $(1,2]$ (max 3 points). Suppose an investor buys the derivative at time $t = 0$. Find the values of p for which the investor expected return for selling the derivative at time $t = 1$ is higher than the expected return at maturity (max 3 points).

Solution: The payoff along the 4 possible paths is

$$Y(u,u) = \left(\frac{1}{3}(8 + 16 + 32) - \frac{29}{3} \right)_+ = 9, \quad Y(u,d) = \left(\frac{1}{3}(8 + 16 + 8) - \frac{29}{3} \right)_+ = 1$$

while $Y(d,u) = Y(d,d) = 0$. By the recurrence formula for European derivatives, with $q_u = 1/3$ and $q_d = 2/3$, we have

$$\Pi_Y(1,u) = e^r (q_u \Pi(2,u,u) + q_d \Pi(2,u,d)) = \frac{1}{3} 9 + \frac{2}{3} 1 = \frac{11}{3}$$

where we used that at maturity the value of the derivative equals the payoff. Similarly we find $\Pi_Y(1,d) = 0$ and

$$\Pi_Y(0) = \frac{1}{3} \Pi_Y(1,u) = \frac{11}{9}.$$

As to the hedging portfolio, we clearly have $h_S(2,d) = 0$, while

$$h_S(2,u) = \frac{1}{S(1,u)} \frac{\Pi(2,u,u) - \Pi(2,u,d)}{e^u - e^d} = \frac{1}{16} \frac{9-1}{2 - 1/2} = \frac{1}{3}, \quad h_S(1) = \frac{1}{8} \frac{11/3}{3/2} = \frac{11}{36}.$$

This answers the first question (3 points). The expected return at maturity is $\mathbb{E}[R_2] = \mathbb{E}[Y] - \Pi_Y(0) = 9p^2 + p(1-p) - \Pi_0$, while the expected return for selling the derivative at time $t = 1$ is $\mathbb{E}[R_1] = \mathbb{E}[\Pi_Y(1)] - \Pi_Y(0) = (11/3)p - \Pi_0$. Hence $\mathbb{E}[R_1] > \mathbb{E}[R_2]$ if and only if

$$9p^2 + p(1-p) - (11/3)p < 0$$

that is $p \in (0, 1/3)$.

3. Let $\delta > 0$. Compute the Black-Scholes price and the position on the stock in the replicating portfolio of the European derivative with payoff $Y = \max(S(T)^\delta, S(T)^{-\delta})$ and maturity T (max 6 points).

Solution: We have $\Pi_Y(t) = v(t, S(t))$, where

$$v(t,x) = e^{-r\tau} \int_{\mathbb{R}} g(xe^{(r-\frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau}y}) e^{-\frac{1}{2}y^2} \frac{dy}{\sqrt{2\pi}}$$

where $\tau = T - t$ and g is the payoff function of the derivative. In this case we have $g(z) = \max(z^\delta, z^{-\delta})$, which equals z^δ is $z > 1$ and $z^{-\delta}$ if $z \leq 1$. Since

$$xe^{(r-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}y} < 1 \Leftrightarrow y < \frac{-\log x - (r - \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}} := d$$

then, dividing the integral domain as $\mathbb{R} = (-\infty, d] \cup (d, \infty)$, we find

$$\begin{aligned} v(t,x) &= e^{-r\tau} x^{-\delta} e^{-\delta(r-\frac{\sigma^2}{2})\tau} \int_{-\infty}^{d} e^{-\delta\sigma\sqrt{\tau}y - \frac{1}{2}y^2} \frac{dy}{\sqrt{2\pi}} + e^{-r\tau} x^{\delta} e^{\delta(r-\frac{\sigma^2}{2})\tau} \int_{d}^{\infty} e^{\delta\sigma\sqrt{\tau}y - \frac{1}{2}y^2} \frac{dy}{\sqrt{2\pi}} \\ &= I_1 + I_2. \end{aligned}$$

Computing the integral in I_1 we obtain

$$I_1 = e^{-(1+\delta)(r-\delta\frac{\sigma^2}{2})\tau} x^{-\delta} \Phi(d + \delta\sigma\sqrt{\tau})$$

In I_2 we replace y with $-y$ in the integral and so doing we find that I_2 is obtained by I_1 with the substitutions $d \to -d$ and $\delta \to -\delta$. Hence

$$I_2 = e^{-(1-\delta)(r+\delta\frac{\sigma^2}{2})\tau} x^{\delta} \Phi(-d - \delta\sigma\sqrt{\tau}).$$

Thus

$$v(t,x) = e^{-(1+\delta)(r-\delta\frac{\sigma^2}{2})\tau} x^{-\delta} \Phi(d_*) + e^{-(1-\delta)(r+\delta\frac{\sigma^2}{2})\tau} x^{\delta} \Phi(-d_*)$$

where

$$d_* = d + \delta\sigma\sqrt{\tau}.$$

The number of stock shares in the replicating portfolio is given by $h_S(t) = \Delta(t, S(t))$, where $\Delta(t,x) = \partial_x v(t,x)$. Using $\Phi' = \phi$ and

$$\partial_x d_* = \partial_x d = -\frac{1}{\sigma\sqrt{\tau}x}$$

we find

$$\Delta(t,x) = e^{-(1+\delta)(r-\delta\frac{\sigma^2}{2})\tau} x^{-\delta} (-\delta\Phi(d_*) - \frac{\phi(d_*)}{\sigma\sqrt{\tau}}) + e^{-(1-\delta)(r+\delta\frac{\sigma^2}{2})\tau} x^{\delta} (\delta\Phi(-d_*) + \frac{\phi(-d_*)}{\sigma\sqrt{\tau}})$$

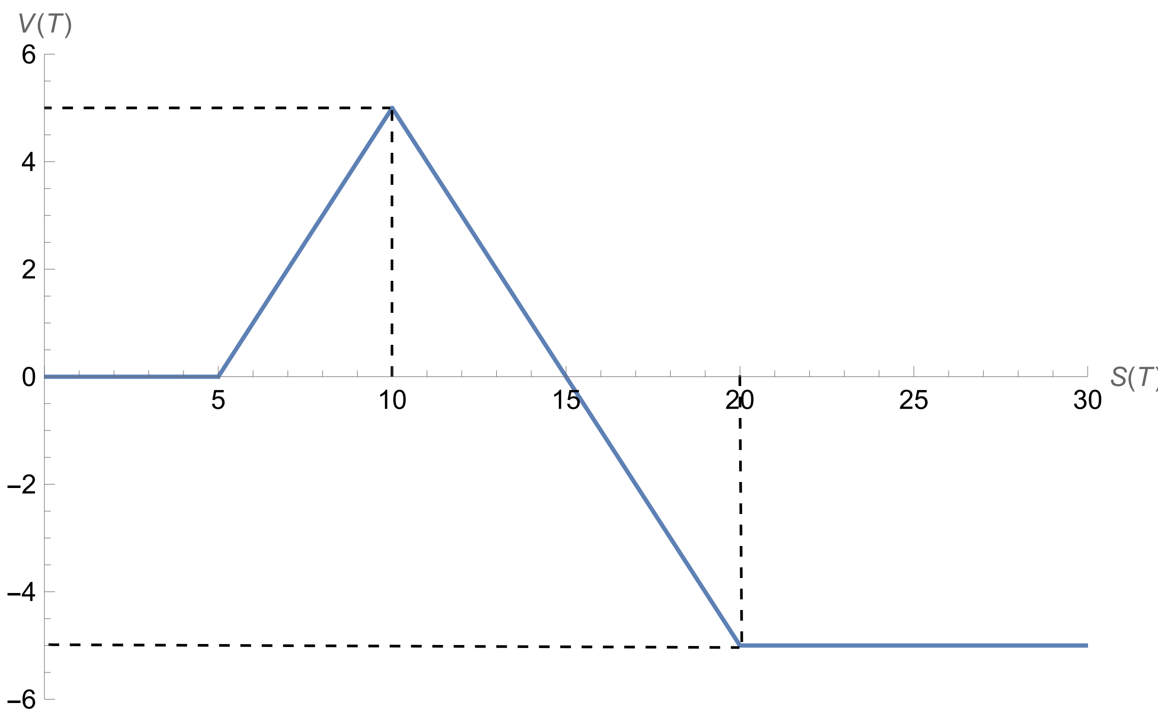

Figure 1: For $S(T) > 20$ the payoff is constant