Exam for the course "Options and Mathematics" (CTH[*MVE095*], GU[*MMG810*]) 2021/22

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REMARKS: (1) NO aids permitted (2) Write as clear as possible: if some step is not clearly readable it will be assumed to be wrong.

Part I

1. Let $V(t)$ be the value at time t of a self-financing portfolio process in a N-period binomial market. Derive the recurrence formula expressing $V(t)$ in terms of $V(t+1)$ (max 3 points). Derive the formula expressing $V(t)$ in terms of $V(N)$ (max 3 points).
2. Give and explain the definition of risk-neutral price of Zero Coupon Bonds (max 3 points).
3. Decide whether the following statements are true or false in an arbitrage free market and explain your answer (max 3 points):
   (a) All risk-free assets have the same interest rate.
   (b) Zero Coupon bonds may have a negative yield in the interval $[0, T]$ even if the risk-free rate at $t = 0$ is positive.
   (c) Investors cannot know at time $t = 0$ if an investment yields a positive return in the interval $[0, T]$.

Solution: (a) True, otherwise an arbitrage opportunity would arise by taking a long position on the risk-free asset with the higher interest rate and a short position, for the same value, on the risk-free asset with the lowest interest rate. (b) True; the ZCB fixes the interest rate to borrow in the interval $[0, T]$. Even if the risk-free rate is positive at $t = 0$ there is no guarantee that will remain so in the whole interval $[0, T]$. (c) True, e.g., a ZCB with positive yield to maturity (which is known at $t = 0$).

Part II

1. An investor wants to set-up a constant portfolio on European stock options in the interval $[0, T]$ such that the pay-off $V(T)$ of the portfolio is one if the stock price lies in the intervals $[1, 2]$ or $[3, 4]$ and is zero otherwise. Give an example for such portfolio (max 3 points). Compute also the Black-Scholes value of this portfolio at time zero, expressed in terms of the standard normal distribution (max 3 points).

Solution: Letting $S(T)$ be the price of the stock at time T, the pay-off of the portfolio can be written as

$$V(T) = H(S(T)-1) - H(S(T)-2) + H(S(T)-3) - H(S(T)-4),$$

where H is the Heaviside function. Hence the sought portfolio is replicated by a long position on one share of the cash settled digital options with strikes 1 and 3 and a short position on one share of the same option with strikes 2 and 4, in each case the notional value being equal to 1. Letting $v(x, K)$ be the Black-Scholes pricing function at $t = 0$ of the cash-settled digital option with strike K, maturity T and notional value one, we then have

$$V(0) = v(S(0), 1) - v(S(0), 2) + v(S(0), 3) - v(S(0), 4).$$

We compute $v(x, K)$ with the Black-Scholes formula:

$$\begin{aligned} v(x, K) &= \frac{e^{-rT}}{\sqrt{2\pi}} \int_{\mathbb{R}} H(xe^{(r-\frac{\sigma^2}{2})T} e^{\sigma\sqrt{T}\,y} - K) e^{-\frac{y^2}{2}}\, dy \\ &= \frac{e^{-rT}}{\sqrt{2\pi}} \int_{-d_{(-)}(x,K)}^{\infty} e^{-\frac{y^2}{2}}\, dy, \end{aligned}$$

where $d_{(-)}(x, K) = \frac{1}{\sigma\sqrt{T}}[\log(x/K) + (r - \frac{\sigma^2}{2})T]$. With the change of variable $y \to -y$ we obtain

$$v(x, K) = e^{-rT}\Phi(d_{(-)}(x, K)).$$

Hence

$$V(0) = e^{-rT}[\Phi(d_{(-)}(S(0), 1)) - \Phi(d_{(-)}(S(0), 2)) + \Phi(d_{(-)}(S(0), 3)) - \Phi(d_{(-)}(S(0), 4))]$$

2. Consider a 3-period binomial market with risk-neutral probability $q = 1/2$, risk-free rate $r = 0$ and physical probability $p \in (0, 1)$. A self-financing portfolio in this market has the following values at $t = 3$:

$$V(3, u, u, u) = -4, \quad V(3, u, u, d) = 0, \quad V(3, u, d, d) = 12, \quad V(3, d, d, d) = -8.$$

Compute the portfolio value $V(t)$ for all $t = 0, 1, 2$ (max 2 points). Study how the expected return of the portfolio depends on p; in particular show that there are two values p_1, p_2 of the physical probability such that the expected return of the portfolio in the interval $[0, 3]$ is zero (max 3 points). In which of the two cases is the portfolio risk lower? (justify your answer without doing any calculation) (max 1 point)

Solution: Using the recurrence formula (with $r = 0$):

$$V(t) = qV^u(t + 1) + (1 - q)V^d(t + 1)$$

we find easily

$$V(2, u, u) = -2,\ V(2, u, d) = V(2, d, u) = 6,\ V(2, d, d) = 2,\ V(1, u) = 2,\ V(1, d) = 4,\ V(0) = 3.$$

Hence the return in the interval $[0,3]$ is the random variable

$$R = \begin{cases} -7 & \text{with prob. } p^3 \\ -3 & \text{with prob. } 3p^2(1-p) \\ 9 & \text{with prob. } 3p(1-p)^2 \\ -11 & \text{with prob. } (1-p)^3 \end{cases}$$

It follows that

$$\mathbb{E}[R] = 4(10p^3 - 24p^2 + 15p) - 11 = f(p).$$

Notice that $f(1/2) = 0$. Moreover

$$f'(p) = 12(10p^2 - 16p + 5) \Rightarrow f'(1/2) = -6$$

hence $f(p) > 0$ for p smaller but close to $1/2$. As $f(0) = -11$ and $f(1) = -7$, then there exists $p_1 \in (0, 1/2)$ such that $f(p) < 0$ for $p \in (0, p_1)$ and $p \in (1/2, 1)$, $f(p) > 0$ for $p \in (p_1, 1/2)$ and $f(p) = 0$ for $p = p_1$ and $p = 1/2$. In particular, the expected return as a (positive) maximum at some $p_* \in (p_1, 1/2)$. The portfolio risk is measured by $\mathrm{Var}(R)$, i.e., by the "randomness" of the return. Now, for $p = p_1$ the portfolio value has a tendency to move down and since the variations of the portfolio return are larger on these paths, then the risk of the portfolio is higher for $p = p_1$ than for $p = 1/2$, even though in both cases the expected return is zero.

3. Let $K > 0$, $T > 0$ and $\{0 = t_0 < t_1 < \cdots < t_n = T\}$ be a uniform partition of the interval $[0, T]$ with size $h = t_i - t_{i-1}$. Assume $r = 0$ and let $t \in [0, T)$ be the present time. Compute the Black-Scholes price $\Pi_Y(t)$ of the European derivative with maturity T and pay-off

$$Y = \left(\sum_{i=1}^{n} S(t_i)h\right) - K$$

as well as the value of K such that this contract is cost-free at time $t = 0$ (max 6 points).

Solution: The answer depends on which interval of the partition lies the present time t. Assume $t \in [t_{j-1}, t_j)$, where $j \in \{1, \dots, n\}$ is given. Then all stock prices up to time t_{j-1} are known, while the prices $t_j, \dots, t_n = T$ are stochastic. Hence if we write the pay-off as

$$Y = \sum_{i=1}^{j-1} S(t_i)h + \sum_{i=j}^{n} S(t_i)h - K$$

we have

$$\Pi_Y(t) = \mathbb{E}_q\left[\sum_{i=j}^{n} S(t_i)h\right] - K + \sum_{i=1}^{j-1} S(t_i)h = h\sum_{i=j}^{n} \mathbb{E}_q[S(t_i)] - K + \sum_{i=1}^{j-1} S(t_i)h$$

As the discounted stock price is a martingale in the risk-neutral probability and $r = 0$, then $\mathbb{E}[S(t_i)] = S(t)$ (the stock price has constant expectation), hence

$$\Pi_Y(t) = S(t)h(n - j + 1) - K + h\sum_{i=1}^{j-1} S(t_i) = S(t)(T - t_{j-1}) - K + h\sum_{i=1}^{j-1} S(t_i)$$

In particular for $t = 0$ we set $j = 1$ and thus

$$\Pi_Y(0) = S(0)T - K.$$

Hence $\Pi_Y(0) = 0$ if and only if $K = S(0)T$.