Exam for the course “Options and Mathematics”
(CTH[MVE095], GU[MMG810]) 2021/22

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January 13th, 2022 (8.30-12.30)

REMARKS: (1) NO aids permitted (2) Write as clear as possible: if some step is not clearly readable it will be assumed to be wrong.

Part I

1. Prove that in a 1-period binomial market there exists a self-financing arbitrage portfolio if and only if \( r \notin (d, u) \) (max 3 points). Prove that the condition \( r \in (d, u) \) is equivalent to the existence of a martingale probability in the \( N \)-period binomial market (max 3 points).

2. Give and explain the definition of self-financing arbitrage portfolio in the binomial market (max 3 points).

3. Assume that the market is arbitrage-free. Let \( P(t, S(t), K, T) \) be the price at time \( t \in [0, T] \) of the European put with strike \( K \) and maturity \( T \) and \( \hat{P}(t, S(t), K, T) \) be the price of the corresponding American put. Assume that the risk-free rate \( r \) is negative and that the underlying stock pays no dividend in the interval \([0, T]\). Decide whether the following statements are true or false and explain your answer (max 3 points):

   (a) It is never optimal to exercise the American derivative prior to maturity.
   (b) \( P(0, S(0), K, T) \) is no greater than \( K \).
   (c) \( \hat{P}(0, S(0), K, T) = P(0, S(0), K, T) \).

**Solution.** (a) True. By the put call-parity,

\[
\hat{P}(t, S(t), K, T) \geq P(t, S(t), K, T) = C(t, S(t), K, T) + Ke^{-r(T-t)} - S(t) > K - S(t),
\]

hence when the American put is in the money we have \( P(t, S(t), K, T) > (K - S(t))_+ \), thus there is no optimal exercise time \( t < T. \) (b) False. Again by the put-call parity, and using \( C(0, S(0), K, T) \to 0 \) as \( S(0) \to 0 \), we have \( P(0, S(0), K, T) \to Ke^{-rT} > K \) as \( S(0) \to 0. \) Hence when the stock price is very small, the put option may be more expensive than the maximum pay-off \( K. \) (c) is true because of (a).

Part II
1. Let \( S(t) > 0 \) be the price at time \( t \) of a non-dividend paying stock. At \( t = 0 \) an investor wants to open a constant portfolio \( A \) on European calls on the stock such that the portfolio value at time \( T \) is

\[
V(T) = \min((S(T) - 2)_+, \frac{1}{2} H(S(T) - 1), (5 - S(T))_+),
\]

where \( H \) is the Heaviside function. Find \( A \) (max. 3 points). Assuming that the stock price follows a geometric Brownian motion with mean of log-return \( \alpha \) and volatility \( \sigma \), find the probability that \( V(T) > 0 \); express the latter result in terms of the standard normal distribution (max. 3 points).

**Solution.** The graph of \( V(T) \) as a function of \( S(T) \) is given as in the following picture

![Graph](image)

by which we can derive that \( A = (C(2), -C(5/2), -C(9/2), C(5)) \), where \( C(K) \) is the call option with strike \( K \) and maturity \( T \). This concludes the first part of the exercise (3 points). Moreover

\[
\mathbb{P}(V(T) > 0) = \mathbb{P}(2 < S(T) < 5) = \mathbb{P}(2 < S(0)e^{\alpha T + \sigma W(T)} < 5) = \mathbb{P}(\Gamma(2) < \frac{W(T)}{\sqrt{T}} < \Gamma(5)),
\]

where

\[
\Gamma(a) = \frac{\log \frac{a}{S(0)} - \alpha T}{\sigma \sqrt{T}}
\]

Using that \( W(T)/\sqrt{T} \in \mathcal{N}(0, 1) \) we find

\[
\mathbb{P}(V(T) > 0) = \int_{\Gamma(2)}^{\Gamma(5)} e^{-\frac{1}{2}y^2} \frac{dy}{\sqrt{2\pi}} = \Phi(\Gamma(5)) - \Phi(\Gamma(2))
\]

where \( \Phi \) is the standard normal distribution.
2. Let $T_2 > T_1$. A chooser option with maturity $T_1$ is a contract that gives the owner the right to obtain at time $T_1$ a European call or a European put expiring at time $T_2$ with no extra cost. Assume $T_1 = 2$, $T_2 = 3$ and that the underlying stock of the options follows a 3-period binomial model with parameters

$$S(0) = 64, \quad u = \log(5/4), \quad d = \log(1/2), \quad r = 0, \quad p = 1/2.$$ 

Assume that the strike of the call is $K = 23$, while the put option is at the money at time $t = 1$. Compute the price of the chooser option at $t = 0$ (max 4 points) and the probability of positive return for the owner of the chooser option in the interval $[0, 3]$ (max 2 points).

**Solution.** The binomial tree of the stock price is given by

$$S(3) = 125 \quad u \quad S(2) = 100 \quad d$$

$$S(1) = 80 \quad u \quad d$$

$$S(0) = 64 \quad \quad \quad \quad \quad \quad S(2) = 40 \quad \quad \quad \quad \quad \quad S(3) = 50$$

$$u \quad d$$

$$S(1) = 32 \quad u \quad d$$

$$S(2) = 16 \quad d$$

The pay-off of the chooser option at time of maturity $t = T_1 = 2$ is $Y = \max(C(2), P(2))$, where $C(t), P(t)$ are the values of the call/put option at time $t$. To compute $C(2)$ we use the recurrence formula for the price of European derivatives:

$$C(2) = e^r (q_u C^u(3) + q_d C^d(3)) = \frac{2}{3} C^u(3) + \frac{1}{3} C^d(3),$$

where we used that $r = 0$ and $q_u = \frac{e^r - e^d}{e^u - e^d} = 2/3$, $q_d = 1 - q_u = 1/3$. Hence, since $C(3) = (S(3) - 23)_+$,

$$C(2, u, u) = \frac{2}{3} C(3, u, u, u) + \frac{1}{3} C(3, u, u, d) = \frac{2}{3} (125 - 23) + \frac{1}{3} (50 - 23) = 77,$$

and similarly

$$C(2, u, d) = C(2, d, u) = 18, \quad C(d, d) = 0.$$
As the put option is at the money at \( t = 1 \), then the strike of the put is \( K = 80 \) if the stock price goes up in the first step and \( K = 32 \) if the stock price goes down in the first step. Hence

\[
P(2, u, u) = \frac{1}{3} P(3, u, u, d) = \frac{1}{3}(80 - 50) = 10,
\]

\[
P(2, u, d) = \frac{2}{3} P(3, u, d, u) + \frac{1}{3} P(3, u, d, d) = \frac{2}{3}(80 - 50) + \frac{1}{3}(80 - 20) = 40,
\]

\[
P(2, d, u) = \frac{1}{3} P(3, d, u, d) = \frac{1}{3}(32 - 20) = 4,
\]

\[
P(2, d, d) = \frac{2}{3} P(3, d, d, u) + \frac{1}{3} P(3, d, d, d) = \frac{2}{3}(32 - 20) + \frac{1}{3}(32 - 8) = 16.
\]

Thus the pay-off of the chooser option at maturity \( T_1 = 2 \) is

\[
Y(u, u) = C(2, u, u) = 77, \quad Y(u, d) = P(2, u, d) = 40,
\]

\[
Y(d, u) = C(2, d, u) = 18, \quad Y(d, d) = P(2, d, d) = 16.
\]

Notice that this chooser option is a non-standard derivative. The price \( \Pi(0) \) of the chooser option at \( t = 0 \) is

\[
\Pi(0) = e^{-2r} \mathbb{E}_q[Y] = \mathbb{E}_q[Y] = (q_u)^2 Y(u, u) + q_u q_d (Y(u, d) + Y(d, u)) + (q_d)^2 Y(d, d) = \frac{440}{9}.
\]

The pay-off at \( t = 3 \) for the owner of the option equals the pay-off of the call along the paths \((u, u, u), (u, u, d), (d, u, u), (d, u, d)\) and the pay-off of the put along the other paths \((u, d, u), (u, d, d), (d, d, u), (d, d, d)\). Hence

\[
R(u, u, u) = 102 - \frac{440}{9} > 0, \quad R(3, u, u, u) = 27 - 440/9 < 0, \quad R(3, d, u, d) < 0
\]

\[
R(u, d, u) = 30 - \frac{440}{9} < 0, \quad R(u, d, d) = 60 - \frac{440}{9} > 0, \quad R(d, d, u) = 12 - \frac{440}{9} < 0, \quad R(d, d, d) < 0
\]

hence \( \mathbb{P}(R > 0) = \mathbb{P}(S^{(u,u,u)}) + \mathbb{P}(S^{(u,d,d)}) = p^3 + p(1-p)^2 = 1/4. \)

3. Let \( K > 0, \quad T > 0 \) and \( t \in [0, T] \). Find the Black-Scholes price \( \Pi_Y(t) \) of the European derivative with maturity \( T \) and pay-off \( Y = (\log S(T) - K)_+ \) and the number of stock shares \( h_S(t) \) in the hedging portfolio for this derivative (max. 3 + 3 points).

**Solution.** We have \( \Pi_Y(t) = v(t, S(t)) \), where the pricing function \( v(t, x) \) is given by

\[
v(t, x) = e^{-rt} \int \mathbb{R} g \left( x e^{(r - \frac{\sigma^2}{2}) \tau + \sigma \sqrt{\tau} y} \right) e^{-\frac{1}{2} y^2} dy, 
\]

where \( \tau = T - t \) is the time left to maturity at time \( t \). Using \( g(z) = (\log z - K)_+ \) we find

\[
g \left( x e^{(r - \frac{\sigma^2}{2}) \tau + \sigma \sqrt{\tau} y} \right) = \begin{cases} 
\log x + (r - \frac{\sigma^2}{2}) \tau + \sigma \sqrt{\tau} y - K & \text{if } y > -d \\
0 & \text{otherwise}
\end{cases}
\]

where

\[
d = \frac{\log x - K + (r - \frac{\sigma^2}{2}) \tau}{\sigma \sqrt{\tau}}.
\]
Replacing in the formula for $v(t, x)$ and computing the integral we find

$$v(t, x) = e^{-rt} \left[ \log x - K + (r - \frac{\sigma^2}{2})t \right] \Phi(d) + \sigma \sqrt{t} e^{-\frac{1}{2}d^2} \left( \frac{e^{-\frac{1}{2}d^2}}{\sqrt{2\pi}} \right) = e^{-rt} \sigma \sqrt{t} (d\Phi(d) + \phi(d))$$

where $\Phi$ is the standard normal distribution and $\phi(d) = \Phi'(d)$ is the standard normal density. This concludes the first part of the exercise (3 points). The number of shares $h_S(t)$ in the hedging portfolio is given by $h_S(t) = \Delta(t, S(t))$, where

$$\Delta(t, x) = \partial_x v(t, x) = e^{-rt} \sigma \sqrt{t} \frac{\partial}{\partial d} (d\Phi(d) + \phi(d)) \frac{\partial d}{\partial x} = e^{-rt} \frac{\Phi(d) + d\phi(d) + \phi'(d)}{x} = e^{-rt} \frac{\Phi(d)}{x}.$$

This concludes the second part of the exercise (3 points).