Exam for the course "Options and Mathematics" (CTH[*MVE095*], GU[*MMG810*]) 2021/22

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January 13th, 2022 (8.30-12.30)

REMARKS: (1) NO aids permitted (2) Write as clear as possible: if some step is not clearly readable it will be assumed to be wrong.

Part I

1. Prove that in a 1-period binomial market there exists a self-financing arbitrage portfolio if and only if $r \notin (d, u)$ (max 3 points). Prove that the condition $r \in (d, u)$ is equivalent to the existence of a martingale probability in the N-period binomial market (max 3 points).

2. Give and explain the definition of self-financing arbitrage portfolio in the binomial market (max 3 points).

3. Assume that the market is arbitrage-free. Let $P(t, S(t), K, T)$ be the price at time $t \in [0, T]$ of the European put with strike K and maturity T and $\widehat{P}(t, S(t), K, T)$ be the price of the corresponding American put. Assume that the risk-free rate r is negative and that the underlying stock pays no dividend in the interval $[0, T]$. Decide whether the following statements are true or false and explain your answer (max 3 points):

   (a) It is never optimal to exercise the American derivative prior to maturity.

   (b) $P(0, S(0), K, T)$ is no greater than K.

   (c) $\widehat{P}(0, S(0), K, T) = P(0, S(0), K, T)$.

   Solution. (a) True. By the put call-parity,

   $$\widehat{P}(t, S(t), K, T) \geq P(t, S(t), K, T) = C(t, S(t), K, T) + Ke^{-r(T-t)} - S(t) > K - S(t),$$

   hence when the American put is in the money we have $P(t, S(t), K, T) > (K - S(t))_+$, thus there is no optimal exercise time $t < T$. (b) False. Again by the put-call parity, and using $C(0, S(0), K, T) \to 0$ as $S(0) \to 0$, we have $P(0, S(0), K, T) \to Ke^{-rT} > K$ as $S(0) \to 0$. Hence when the stock price is very small, the put option may be more expensive than the maximum pay-off K. (c) is true because of (a).

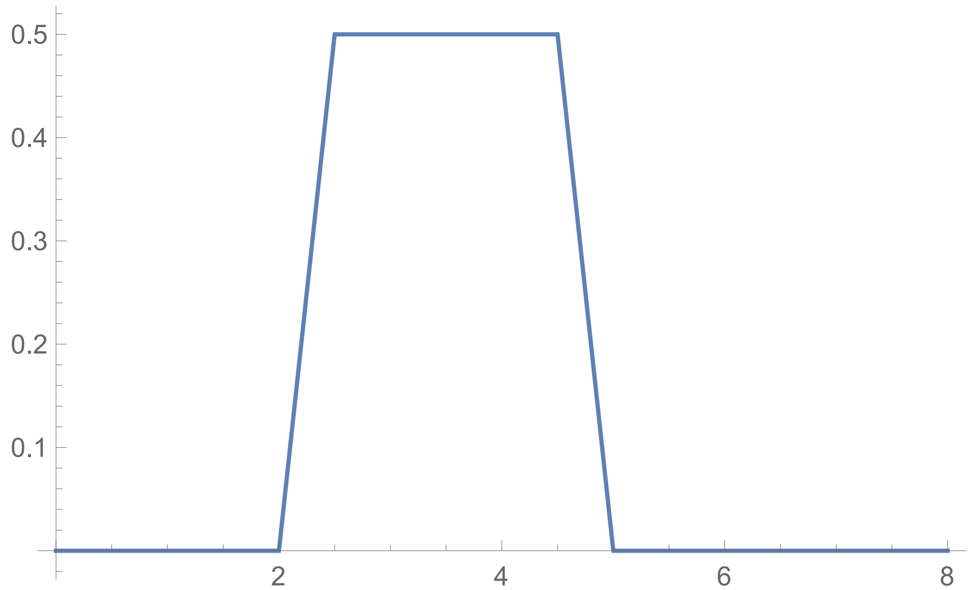
Part II

1. Let $S(t) > 0$ be the price at time t of a non-dividend paying stock. At $t = 0$ an investor wants to open a constant portfolio $\mathcal{A}$ on European calls on the stock such that the portfolio value at time T is

$$V(T) = \min((S(T) - 2)_+, \frac{1}{2}H(S(T) - 1), (5 - S(T))_+),$$

where H is the Heaviside function. Find $\mathcal{A}$ (max. 3 points). Assuming that the stock price follows a geometric Brownian motion with mean of log-return α and volatiliy σ, find the probability that $V(T) > 0$; express the latter result in terms of the standard normal distribution (max. 3 points).

Solution. The graph of $V(T)$ as a function of $S(T)$ is given as in the following picture



by which we can derive that $\mathcal{A} = (C(2), -C(5/2), -C(9/2), C(5))$, where $C(K)$ is the call option with strike K and maturity T. This concludes the first part of the exercise (3 points). Moreover

$$\mathbb{P}(V(T) > 0) = \mathbb{P}(2 < S(T) < 5) = \mathbb{P}(2 < S(0)e^{\alpha T + \sigma W(T)} < 5) = \mathbb{P}(\Gamma(2) < \frac{W(T)}{\sqrt{T}} < \Gamma(5)),$$

where

$$\Gamma(a) = \frac{\log \frac{a}{S(0)} - \alpha T}{\sigma\sqrt{T}}$$

Using that $W(T)/\sqrt{T} \in \mathcal{N}(0, 1)$ we find

$$\mathbb{P}(V(T) > 0) = \int_{\Gamma(2)}^{\Gamma(5)} e^{-\frac{1}{2}y^2} \frac{dy}{\sqrt{2\pi}} = \Phi(\Gamma(5)) - \Phi(\Gamma(2))$$

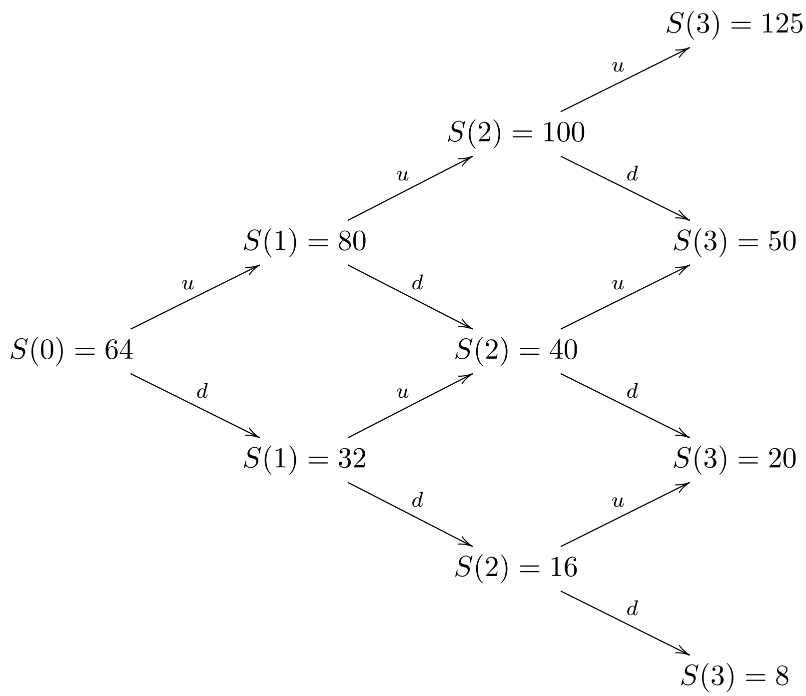
where Φ is the standard normal distribution.

2. Let $T_2 > T_1$. A chooser option with maturity T_1 is a contract that gives the owner the right to obtain at time T_1 a European call or a European put expiring at time T_2 with no extra cost. Assume $T_1 = 2$, $T_2 = 3$ and that the underlying stock of the options follows a 3-period binomial model with parameters

$$S(0) = 64, \quad u = \log(5/4), \quad d = \log(1/2), \quad r = 0, \quad p = 1/2.$$

Assume that the strike of the call is $K = 23$, while the put option is at the money at time $t = 1$. Compute the price of the chooser option at $t = 0$ (max 4 points) and the probability of positive return for the owner of the chooser option in the interval $[0, 3]$ (max 2 points).

Solution. The binomial tree of the stock price is given by



The pay-off of the chooser option at time of maturity $t = T_1 = 2$ is $Y = \max(C(2), P(2))$, where $C(t), P(t)$ are the values of the call/put option at time t. To compute $C(2)$ we use the recurrence formula for the price of European derivatives:

$$C(2) = e^r(q_u C^u(3) + q_d C^d(3)) = \frac{2}{3}C^u(3) + \frac{1}{3}C^d(3),$$

where we used that $r = 0$ and $q_u = \frac{e^r - e^d}{e^u - e^d} = 2/3$, $q_d = 1 - q_u = 1/3$. Hence, since $C(3) = (S(3) - 23)_+$,

$$C(2, u, u) = \frac{2}{3}C(3, u, u, u) + \frac{1}{3}C(3, u, u, d) = \frac{2}{3}(125 - 23) + \frac{1}{3}(50 - 23) = 77,$$

and similarly

$$C(2, u, d) = C(2, d, u) = 18, \quad C(d, d) = 0.$$

As the put option is at the money at $t = 1$, then the strike of the put is $K = 80$ if the stock price goes up in the first step and $K = 32$ if the stock price goes down in the first step. Hence

$$P(2, u, u) = \frac{1}{3}P(3, u, u, d) = \frac{1}{3}(80 - 50) = 10,$$

$$P(2, u, d) = \frac{2}{3}P(3, u, d, u) + \frac{1}{3}P(3, u, d, d) = \frac{2}{3}(80 - 50) + \frac{1}{3}(80 - 20) = 40,$$

$$P(2, d, u) = \frac{1}{3}P(3, d, u, d) = \frac{1}{3}(32 - 20) = 4,$$

$$P(2, d, d) = \frac{2}{3}P(3, d, d, u) + \frac{1}{3}P(3, d, d, d) = \frac{2}{3}(32 - 20) + \frac{1}{3}(32 - 8) = 16.$$

Thus the pay-off of the chooser option at maturity $T_1 = 2$ is

$$Y(u, u) = C(2, u, u) = 77, \quad Y(u, d) = P(2, u, d) = 40,$$

$$Y(d, u) = C(2, d, u) = 18, \quad Y(d, d) = P(2, d, d) = 16.$$

Notice that this chooser option is a non-standard derivative. The price $\Pi(0)$ of the chooser option at $t = 0$ is

$$\Pi(0) = e^{-2r}\mathbb{E}_q[Y] = \mathbb{E}_q[Y] = (q_u)^2 Y(u, u) + q_u q_d(Y(u, d) + Y(d, u)) + (q_d)^2 Y(d, d) = \frac{440}{9}.$$

The pay-off at $t = 3$ for the owner of the option equals the pay-off of the call along the paths $(u, u, u), (u, u, d), (d, u, u), (d, u, d)$ and the pay-off of the put along the other paths $(u, d, u), (u, d, d), (d, d, u), (d, d, d)$. Hence

$$R(u, u, u) = 102 - \frac{440}{9} > 0, \quad R(3, u, u, d) = R(3, d, u, u) = 27 - 440/9 < 0, \quad R(3, d, u, d) < 0$$

$$R(u, d, u) = 30 - \frac{440}{9} < 0, \quad R(u, d, d) = 60 - \frac{440}{9} > 0, \quad R(d, d, u) = 12 - \frac{440}{9} < 0, \quad R(d, d, d) < 0$$

hence $\mathbb{P}(R > 0) = \mathbb{P}(S^{(u,u,u)}) + \mathbb{P}(S^{(u,d,d)}) = p^3 + p(1 - p)^2 = 1/4$.

3. Let $K > 0$, $T > 0$ and $t \in [0, T]$. Find the Black-Scholes price $\Pi_Y(t)$ of the European derivative with maturity T and pay-off $Y = (\log S(T) - K)_+$ and the number of stock shares $h_S(t)$ in the hedging portfolio for this derivative (max. 3 + 3 points).

   Solution. We have $\Pi_Y(t) = v(t, S(t))$, where the pricing function $v(t, x)$ is given by

$$v(t, x) = e^{-r\tau} \int_{\mathbb{R}} g\left(xe^{\left(r - \frac{\sigma^2}{2}\right)\tau + \sigma\sqrt{\tau}y}\right) e^{-\frac{1}{2}y^2} \frac{dy}{\sqrt{2\pi}},$$

   where $\tau = T - t$ is the time left to mautiry at time t. Using $g(z) = (\log z - K)_+$ we find

$$g\left(xe^{\left(r - \frac{\sigma^2}{2}\right)\tau + \sigma\sqrt{\tau}y}\right) = \begin{cases} \log x + (r - \frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau}y - K & \text{if } y > -d \\ 0 & \text{otherwise} \end{cases}$$

   where

$$d = \frac{\log x - K + (r - \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}.$$

Replacing in the formula for $v(t,x)$ and computing the integral we find

$$v(t,x) = e^{-r\tau}\left[\left(\log x - K + (r - \frac{\sigma^2}{2})\tau\right)\Phi(d) + \sigma\sqrt{\tau}\frac{e^{-\frac{1}{2}d^2}}{\sqrt{2\pi}}\right] = e^{-r\tau}\sigma\sqrt{\tau}(d\Phi(d) + \phi(d))$$

where Φ is the standard normal distribution and $\phi(d) = \Phi'(d)$ is the standard normal density. This concludes the first part of the exercise (3 points). The number of shares $h_S(t)$ in the hedging portfolio is given by $h_S(t) = \Delta(t, S(t))$, where

$$\Delta(t,x) = \partial_x v(t,x) = e^{-r\tau}\sigma\sqrt{\tau}\frac{\partial}{\partial d}(d\Phi(d)+\phi(d))\frac{\partial d}{\partial x} = e^{-r\tau}\frac{\Phi(d) + d\phi(d) + \phi'(d)}{x} = e^{-r\tau}\frac{\Phi(d)}{x}.$$

This concludes the second part of the exercise (3 points).