Exam for the course "Options and Mathematics" (CTH[*MVE095*], GU[*MMG810*]) 2020/21

For questions call the examiner at +46 (0)31 772 35 62

August 24^{th}, 2021 (8.30-12.30)

REMARKS: (1) All aids permitted, however you must work alone (2) Give all details and explain all steps in your solutions (3) Write as clear as possible: if some step is not clearly readable it will assumed to be wrong! (4) See the course homepage for instructions on how to submit the exam.

Part I

1. Derive the formula for the Black-Scholes price of call options (max 2 points).

2. Give and explain the definition of self-financing portfolio in a binomial market (max 2 points).

3. Assume that the market is arbitrage-free and $r > 0$. Decide whether the following statements are true or false and explain your answer (max 2 points):

   (a) A European put in the money is at any time more valuable than the corresponding European call with the same strike and maturity.

   (b) A European call in the money is at any time more valuable than the corresponding European put with the same strike and maturity.

   Solution. In both cases the correct answer can be derived using the put-call parity: $C - P = S - Ke^{-r\tau}$, where τ is the time left to maturity, C, P, S the prices of the call, put and stock at time t and K the strike price. **(a) is false**. Assuming for instance $S = K/2$, then the put is in the money and $C = P + K(1/2 - e^{-r\tau}) > P$ if τ is sufficiently large. **(b) is true**. Since the call is in the money, then $S > K$ and therefore $C = P + S - Ke^{r\tau} > P + K(1 - e^{-r\tau}) \geq P$, hence $C > P$.

Part II

1. Find a portfolio on European call, put and digital options that replicates the European derivative with pay-off Y depicted in the figure, see next page (max. 4 points).

   Answer. For instance, $(2C(2), 3P(2), -2C(4), 2H(4), -6H(10))$, where $C(K)$ is the call with strike K, $P(K)$ the put with strike K and $H(K)$ the cash settled digital call with strike K and notional value 1.

2. Consider a European style option in a binomial market with the following pay-off at maturity $T = N$:
$$Y = \sum_{t=1}^{N} (S(t)/S(t-1) - 1)_+.$$

Assuming $u > 0$, $d < 0$ and $r = 0$, compute the price at time $t = 0$ of this derivative (max. 1 point), the probability that it expires in the money (max. 1 point) and the expected return at maturity (max. 2 points).

Solution. We have

$$\frac{S(t)}{S(t-1)} = \begin{cases} e^u & \text{with prob. } p \\ e^d & \text{with prob. } 1-p \end{cases}$$

Hence, since $u > 0$ and $d < 0$,

$$\left(\frac{S(t)}{S(t-1)} - 1\right)_+ = \begin{cases} e^u - 1 & \text{with prob. } p \\ 0 & \text{with prob. } 1-p \end{cases}$$

It follows by the risk-neutral pricing formula that

$$\Pi_Y(0) = e^{-rN}\mathbb{E}_q[Y] = e^{-rN}\sum_{t=1}^{N}\mathbb{E}_q[(S(t)/S(t-1)-1)_+] = e^{-rN}N(e^u-1)q,$$

where $q = (e^r - e^d)/(e^u - e^d) \in (0,1)$ is the risk-neutral probability parameter. This anwers the first question (1 point). The derivative expires always in the money, except if the stock price decreases at each step, hence the probability of positive return is $1-(1-p)^N$, which answers the second question (1 point). To compute the expected return at maturity, we observe that the pay-off along a path x is given by $Y(x) = N_u(x)(e^u - 1)$, where $N_u(x)$ is the number of u in the path x. Hence the expected return at maturity is

$$\mathbb{E}_p[R] = \mathbb{E}_p[Y] - \Pi_Y(0) = \mathbb{E}_p[Y] - N(e^u-1)q = (e^u-1)(\mathbb{E}_p[N_u] - Nq] = (e^u-1)N(p-q),$$

where we used that the expected number of u's is Np. This answers the third question (2 points).

3. Let $0 = t_0 < t_1 < \cdots < t_N = T$, $t_i - t_{i-1} = h$, be a uniform partition of the interval $[0, T]$. Compute the Black-Scholes price at $t = 0$ of the European derivative with pay-off

$$Y = \sum_{i=1}^{N}(S(t_i) - S(t_{i-1}))_+$$

and maturity T (max. 4 points).

Solution. By the risk-neutral pricing formula we have

$$\Pi_Y(0) = e^{-rT}\sum_{i=1}^{N}\mathbb{E}_q[(S(t_i) - S(t_{i-1}))_+].$$

Using the formula for the geometric Brownian motion in the risk-neutral probability we have

$$\begin{aligned} S(t_i) - S(t_{i-1}) &= S(0)e^{\left(r-\frac{\sigma^2}{2}\right)t_i + \sigma W(t_i)} - S(0)e^{\left(r-\frac{\sigma^2}{2}\right)t_{i-1} + \sigma W(t_{i-1})} \\ &= S(0)e^{\left(r-\frac{\sigma^2}{2}\right)(t_{i-1}+h) + \sigma W(t_{i-1}) + \sigma\sqrt{h}G} - S(0)e^{\left(r-\frac{\sigma^2}{2}\right)t_{i-1} + \sigma W(t_{i-1})}, \end{aligned}$$

where $G = (W(t_i) - W(t_{i-1}))/\sqrt{h} \in N(0,1)$. Hence

$$(S(t_i) - S(t_{i-1}))_+ = S(0)e^{\left(r-\frac{\sigma^2}{2}\right)t_{i-1}+\sigma W(t_{i-1})}\left(e^{\left(r-\frac{\sigma^2}{2}\right)h+\sigma\sqrt{h}G} - 1\right)_+$$

Hence, using that $W(t_{i-1})$ and G are independent random variables,

$$\Pi_Y(0) = S(0)e^{-rT}\mathbb{E}_q\left[\left(e^{\left(r-\frac{\sigma^2}{2}\right)h+\sigma\sqrt{h}G} - 1\right)_+\right]\sum_{i=1}^N e^{\left(r-\frac{\sigma^2}{2}\right)t_{i-1}}\mathbb{E}_q\left[e^{\sigma W(t_{i-1})}\right]$$

The expectation outside the sum is $e^{rh}C(0,1,1,h)$. The expectation within the sum is

$$\mathbb{E}_q\left[e^{\sigma W(t_{i-1})}\right] = \int_{\mathbb{R}} e^{\sigma\sqrt{t_{i-1}}\,x-\frac{x^2}{2}}\frac{dx}{\sqrt{2\pi}} = e^{\frac{\sigma^2}{2}(t_{i-1})}.$$

Hence

$$\Pi_Y(0) = S(0)e^{-rT}e^{rh}C(0,1,1,h)\sum_{i=1}^N e^{rt_{i-1}}.$$

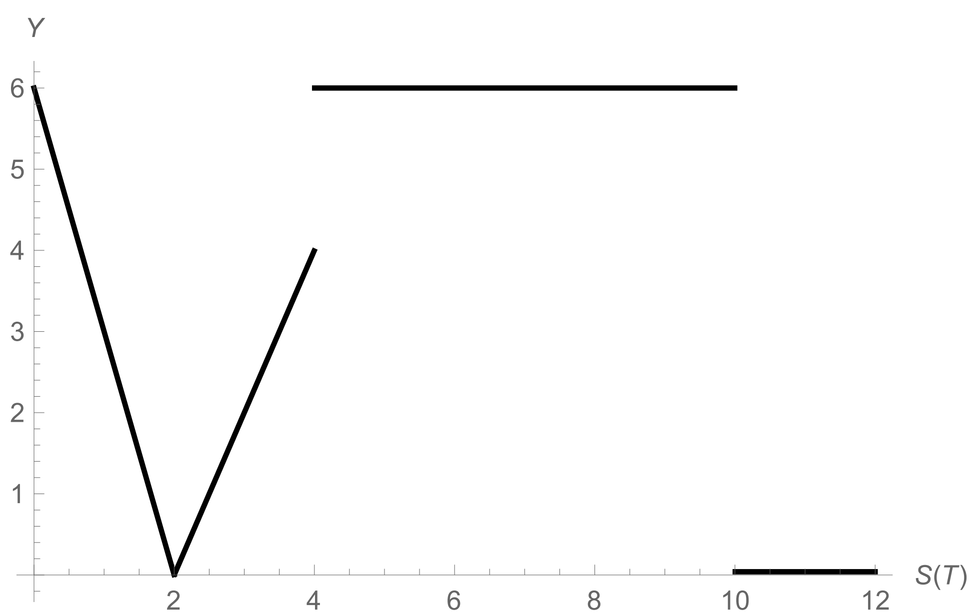
This formula can be further simplified by using that

$$\sum_{i=1}^N e^{rt_{i-1}} = \sum_{i=1}^N e^{r(i-1)h} = \sum_{i=0}^{N-1}(e^{rh})^i = \frac{1-e^{rhN}}{1-e^{rh}} = \frac{1-e^{rT}}{1-e^{rh}}.$$

In conclusion

$$\Pi_Y(0) = \frac{e^{-rT}-1}{e^{-rh}-1}C(0,1,1,h).$$

(This formula also holds in the limit $r \to 0$).



Remark: For $S(T) > 10$ the pay-off is identically zero.