Exam for the course "Options and Mathematics" (CTH[*MVE095*], GU[*MMG810*]) 2018/19

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REMARKS: (1) No aids permitted (2) Minor errors in the calculations will be forgiven, but remember that fractions look nicer when you simplify them!

1. (i) Define and explain the concept of arbitrage portfolio process invested in a binomial market (max. 1 point)

   (ii) Assume that the dominance principle holds. Prove the put-call parity (max. 2 points). Prove that the price of European call options is a convex function of the strike price (max. 2 points).

   Solution. See lecture notes

2. Consider the derivative with intrinsic value

$$Y(t) = \min(S(t), (24 - S(t))_+)$$

and expiring at time $T = 3$. The initial price of the underlying stock is $S(0) = 27$, while at future times it follows the binomial model

$$S(t+1) = \begin{cases} 4S(t)/3 & \text{with probability } 1/2 \\ 2S(t)/3 & \text{with probability } 1/2 \end{cases}$$

for $t = 0, 1, 2$. The interest rate of the money market is $r = \log(7/6)$. Assume that the owner of the derivative is allowed to exercise at time $t = 2$ or at maturity, but *not* at time $t = 1$ (this is an example of Bermuda option). Compute the possible paths of the binomial price $\{\widetilde{\Pi}_Y(t)\}_{t=0,1,2,3}$ of the derivative (max 3 points). In which case it is optimal for the buyer to exercise the derivative prior to expiration (max 1 point)? What is the amount of cash that the seller can withdraw from the portfolio if the buyer does not exercise the derivative optimally? (max 1 point).

Solution. With the given values of the parameters u, d, r, we have

$$q_u = \frac{e^r - e^d}{e^u - e^d} = \frac{\frac{7}{6} - \frac{2}{3}}{\frac{4}{3} - \frac{2}{3}} = \frac{3}{4} \Rightarrow q_d = \frac{1}{4}.$$

The binomial tree of the stock price is

$$
\begin{tikzcd}[column sep=small]
 & & & S(3)=64 \\
 & & S(2)=48 \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & S(1)=36 \arrow[ur, "u"] \arrow[dr, "d"] & & S(3)=32 \\
S(0)=27 \arrow[ur, "u"] \arrow[dr, "d"] & & S(2)=24 \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & S(1)=18 \arrow[ur, "u"] \arrow[dr, "d"] & & S(3)=16 \\
 & & S(2)=12 \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & & & S(3)=8
\end{tikzcd}
$$

to which there corresponds the following diagram for the intrinsic value:

$$
\begin{tikzcd}[column sep=small]
 & & & Y(3)=0=\widetilde{\Pi}_Y(3) \\
 & & Y(2)=0 \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & Y(1)=0 \arrow[ur, "u"] \arrow[dr, "d"] & & Y(3)=0=\widetilde{\Pi}_Y(3) \\
Y(0)=0 \arrow[ur, "u"] \arrow[dr, "d"] & & Y(2)=0 \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & Y(1)=6 \arrow[ur, "u"] \arrow[dr, "d"] & & Y(3)=8=\widetilde{\Pi}_Y(3) \\
 & & Y(2)=12 \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & & & Y(3)=8=\widetilde{\Pi}_Y(3)
\end{tikzcd}
$$

The intrinsic value of this derivative coincides with the pay-off only at the times when the derivative can be exercised. Letting $\widetilde{Y}(t)$ be the pay-off of the derivative exercised at time t

we then have

$$\begin{tikzcd}
 & & & \widetilde{Y}(3)=0=\widetilde{\Pi}_Y(3) \\
 & & \widetilde{Y}(2)=0 \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & \widetilde{Y}(1)=0 \arrow[ur, "u"] \arrow[dr, "d"] & & \widetilde{Y}(3)=0=\widetilde{\Pi}_Y(3) \\
\widetilde{Y}(0)=0 \arrow[ur, "u"] \arrow[dr, "d"] & & \widetilde{Y}(2)=0 \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & \widetilde{Y}(1)=0 \arrow[ur, "u"] \arrow[dr, "d"] & & \widetilde{Y}(3)=8=\widetilde{\Pi}_Y(3) \\
 & & \widetilde{Y}(2)=12 \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & & & \widetilde{Y}(3)=8=\widetilde{\Pi}_Y(3)
\end{tikzcd}$$

since the derivative cannot be exercised at time $t = 1$. The binomial price of the derivative can now be computed using the recurrence formula

$$\widetilde{\Pi}_Y(t) = \max(\widetilde{Y}(t), e^{-r}(q_u\widetilde{\Pi}_Y^u(t+1)+q_d\widetilde{\Pi}_Y^d(t+1))) = \max(\widetilde{Y}(t), \frac{6}{7}(\frac{3}{4}\widetilde{\Pi}_Y^u(t+1)+\frac{1}{4}\widetilde{\Pi}_Y^d(t+1))),$$

and we find

$$\begin{tikzcd}
 & & & \widetilde{\Pi}_Y(3)=0 \\
 & & \widetilde{\Pi}_Y(2)=0 \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & \widetilde{\Pi}_Y(1)=6/7 \arrow[ur, "u"] \arrow[dr, "d"] & & \widetilde{\Pi}_Y(3)=0 \\
\widetilde{\Pi}_Y(0)=81/49 \arrow[ur, "u"] \arrow[dr, "d"] & & \widetilde{\Pi}_Y(2)=4 \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & \widetilde{\Pi}_Y(1)=36/7 \arrow[ur, "u"] \arrow[dr, "d"] & & \widetilde{\Pi}_Y(3)=8 \\
 & & \boxed{\widetilde{\Pi}_Y(2)=12} \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & & & \widetilde{\Pi}_Y(3)=8
\end{tikzcd}$$

This concludes the first part of the exercise (3 points). In the event marked with a box, i.e., when the stock price goes down in the first and second step, the value of the derivative equals the pay-off and thus in this case it is optimal to exercise the derivative (1 point). If the buyer does not exercise the derivative optimally, the writer can withdraw the cash from the portfolio given by

$$C(2) = \widetilde{\Pi}_Y(2) - e^{-r}[q_u \widetilde{\Pi}_Y^u(3) + q_d \widetilde{\Pi}_Y^d(3)] = \widetilde{\Pi}_Y(2) - \frac{6}{7}[\frac{3}{4}\widetilde{\Pi}_Y^u(3) + \frac{1}{4}\widetilde{\Pi}_Y^d(2)] = 48/7 \qquad \text{(1 point).}$$

3. Find the Black-Scholes price (max 2 points) and the hedging portfolio (max 1 point) of the European derivative with pay-off $Y = S(T) + S(T)^{-1}$ at maturity $T > 0$. Compute the expected return for the buyer of the derivative (max 2 points).

   Solution. The pay-off function is

$$g(z) = z + 1/z$$

   and the Black-Scholes price is $\Pi_Y(t) = v(t, S(t))$, where

$$v(t,x) = \frac{e^{-r\tau}}{\sqrt{2\pi}} \int_{\mathbb{R}} g\left(xe^{(r-\frac{\sigma^2}{2})\tau} e^{\sigma\sqrt{\tau}\,y}\right) e^{-\frac{y^2}{2}}\,dy, \quad \tau = T - t.$$

   Computing the integral we find

$$\Pi_Y(t) = S(t) + e^{(\sigma^2-2r)\tau} S(t)^{-1} \quad \text{(2 points).}$$

   The number of stock shares in the hedging portfolio is $h_S(t) = \partial_x v(t, S(t)) = 1 - S(t)^{-2} e^{(\sigma^2-2r)\tau}$ (1 point). The expected return is

$$\begin{aligned} R &= \mathbb{E}[\Pi_Y(T) - \Pi_Y(0)] = \mathbb{E}[Y] - S(0) - e^{(\sigma^2-2r)T} S(0)^{-1} \\ &= \mathbb{E}[S(T)] + \mathbb{E}[S(T)^{-1}] - S(0) - e^{(\sigma^2-2r)T} S(0)^{-1}. \end{aligned}$$

   Now, the stock price is given by the geometric Brownian motion $S(t) = S(0)e^{\alpha t + \sigma W(t)}$, hence

$$\mathbb{E}[S(T)] + \mathbb{E}[S(T)^{-1}] = S(0)e^{\alpha T}\mathbb{E}[e^{\sigma W(T)}] + S(0)^{-1}e^{-\alpha T}\mathbb{E}[e^{-\sigma W(T)}]$$

   Using the normal density of the Brownian motion we find $\mathbb{E}[e^{\sigma W(T)}] = \mathbb{E}[e^{-\sigma W(T)}] = e^{\sigma^2 T/2}$, hence

$$R = S(0)\left(e^{(\alpha+\frac{\sigma^2}{2})T} - 1\right) + S(0)^{-1}\left(e^{(-\alpha+\frac{\sigma^2}{2})T} - e^{(\sigma^2-2r)T}\right).$$

   This concludes the third part of the exercise (2 points).