Exam for the course Options and Mathematics (CTH[MVE095], GU[MMA700]) 2017/18

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REMARKS: No aids permitted

1. Let $\sigma > 0$ be the volatility of the Black-Scholes market and let $r > 0$ be the interest rate of the risk-free asset. Let $P(t)$ be the Black-Scholes price at time t of a European put on a stock with price $S(t)$. Let $K > 0$ and $T > 0$ be the strike and the maturity time of this put, respectively.

(i) Write down the formula for the Black-Scholes price of this European put (max. 1 point). (Note that you do NOT need to prove the formula).

(ii) Compute the following limits:

$$\lim_{\sigma\to 0+} P(t), \quad \lim_{\sigma\to\infty} P(t), \quad \lim_{T\to\infty} P(t), \quad \lim_{K\to\infty} P(t).$$

Each limit gives 1 point if it is correct, 0 otherwise.

Solution: (i) It follows from Theorem 6.5 in Lecture Notes that the Black-Scholes price of a European put is $P(t, S(t), K, T)$. Here,

$$P(t, x, K, T) = \Phi(-d_2)Ke^{-r\tau} - \Phi(-d_1)x,$$

where

$$d_2 = \frac{\log\left(\frac{x}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}, \quad d_1 = d_2 + \sigma\sqrt{\tau},$$

and where $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}y^2}\,dy$ is the standard normal distribution.

(ii) If $\sigma \to 0+$, then d_1 and d_2 have the same limit. We have

$$\lim_{\sigma\to 0+} d_2 = \lim_{\sigma\to 0+} \frac{\log\left(\frac{x}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}} = \lim_{\sigma\to 0+} \frac{\log\left(\frac{x}{K}e^{r\tau}\right)}{\sigma\sqrt{\tau}}.$$

Consider cases:

1. $x > Ke^{r\tau}$. We have $d_1, d_2 \to +\infty$ as $\sigma \to 0+$. Thus, $\Phi(-d_1), \Phi(-d_2) \to 0$ and $P(t) \to 0$ as $\sigma \to 0+$.

2. $x = Ke^{r\tau}$. We have $d_1, d_2 \to 0$ as $\sigma \to 0+$. Thus, $\Phi(-d_1), \Phi(-d_2) \to \Phi(0)$ and, so, $P(t) \to \Phi(0)(Ke^{r\tau} - x) = 0$ as $\sigma \to 0+$.

3. $x < Ke^{r\tau}$. We have $d_1, d_2 \to -\infty$ as $\sigma \to 0+$. Thus, $\Phi(-d_1), \Phi(-d_2) \to 1$ and $P(t) \to Ke^{r\tau} - x$ as $\sigma \to 0+$.

Summing up, $\lim_{\sigma\to 0+} P(t) = (Ke^{r\tau} - x)_+$.

If $\sigma \to \infty$, then $d_2 \to -\infty$ and $d_1 \to +\infty$. Thus, $\Phi(-d_1) \to 0$, $\Phi(-d_2) \to 1$ and $P(t) \to Ke^{r\tau}$ as $\sigma \to \infty$.

If $T \to \infty$, then (recalling that $\tau = T - t$) $d_1, d_2 \to +\infty$ and $P(t) \to 0$ as in Case 1 above.

If $K \to \infty$, then $d_1, d_2 \to -\infty$ and $\Phi(-d_1), \Phi(-d_2) \to 1$. Thus, $P(t) \to +\infty$.

2. Assume that the dominance principle holds.

(i) Prove the put-call parity (max. 2 points).

(ii) Let the interest rate $r \geq 0$. Prove that the price of a European call option cannot be less than its intrinsic value (max. 1 point).

(iii) Let the interest rate $r \geq 0$. Prove that the price of a European call option is a non-decreasing function of the maturity time (max. 2 points).

Solution: see Theorem 1.2 (iv), (v), (vi) in Lecture Notes.

3. Let $T_2 > T_1 > 0$ and $K_1, K_2 > 0$. A **put on call** option with maturity T_1 and strike K_1 is a contract that gives to its owner the right to sell at time T_1 for the price K_1 a call option on a stock with maturity T_2 and strike K_2. Let $S(t)$ be the price of the underlying stock of the call option. Assume that $S(t)$ follows a 2-period binomial model with parameters

$$e^u = \frac{5}{4}, \quad e^d = \frac{3}{4}, \quad e^r = \frac{9}{8}, \quad p = \frac{1}{3}.$$

Assume further that $S(0) = 48$, $T_2 = 2$, $T_1 = 1$, $K_1 = 11$, $K_2 = 36$. Compute the initial price of the put on call (max. 3 points). Compute the probability that the return of a constant portfolio with a long position on this derivative be positive (max. 2 points).

Solution: The binomial tree of the stock price is

$$
\begin{tikzcd}
 & & S(2)=75 \\
 & S(1)=60 \arrow[ur, "u"] \arrow[dr, "d"] & \\
S(0)=48 \arrow[ur, "u"] \arrow[dr, "d"] & & S(2)=45 \\
 & S(1)=36 \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & & S(2)=27
\end{tikzcd}
$$

We have $q_u = 3/4$ and $q_d = 1/4$. First, we compute the price of the underlying call option with maturity $T_2 = 2$ and strike $K_2 = 36$. We have $\Pi_{call}(2) = Y_{call} = (S(2) - K_2)_+$. Using this and the recurrence formula

$$\Pi_{call}(t) = e^{-r}(q_u \Pi^u_{call}(t+1) + q_d \Pi^d_{call}(t+1), \ t = 0, 1,$$

we obtain

$$
\begin{tikzcd}
 & & \Pi_{call}(2)=39 \\
 & \Pi_{call}(1)=28 \arrow[ur, "u"] \arrow[dr, "d"] & \\
\Pi_{call}(0)=20 \arrow[ur, "u"] \arrow[dr, "d"] & & \Pi_{call}(2)=9 \\
 & \Pi_{call}(1)=6 \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & & \Pi_{call}(2)=0
\end{tikzcd}
$$

The price of put on call at time $t = T_1 = 1$ is equal to its pay-off, that is

$\Pi_{pc}(1) = (K_1 - \Pi_{call}(1))_+$. Thus,

$$\begin{tikzcd}[column sep=small]
 & \Pi_{pc}(1) = 0 \\
\Pi_{pc}(0) = \frac{10}{9} \arrow[ur, "u"] \arrow[dr, "d"'] & \\
 & \Pi_{pc}(1) = 5
\end{tikzcd}$$

Consider the constant portfolio with +1 share of the put on call. The return of this portfolio is path dependent. Also, it depends on whether the put was exercised. Since the only optimal time to exercise is $t = 1$ when the stock price goes down, we obtain

$$R(u,u) = -\frac{10}{9} + 0 = -\frac{10}{9}, \quad R(u,d) = -\frac{10}{9} + 0 = -\frac{10}{9},$$

$$R(d,u) = -\frac{10}{9} + 6 - 9 = -\frac{37}{9}, \quad R(d,d) = -\frac{10}{9} + 6 - 0 = \frac{44}{9}.$$

Thus, the expected return is

$$\mathbb{E}(R) = (\frac{1}{3})^2(-\frac{10}{9}) + \frac{1}{3}\frac{2}{3}(-\frac{10}{9}) + \frac{2}{3}\frac{1}{3}(-\frac{37}{9}) + (\frac{2}{3})^2\frac{44}{9} = \frac{8}{9}.$$