1. Consider a 3-period binomial asset pricing model with the following parameters:

\[ e^u = \frac{4}{3}, \quad e^d = \frac{1}{3}, \quad p = \frac{7}{8}. \]

Assuming \( S(0) = 27 \) and that the risk-free asset has zero interest rate \( r = 0 \),
compute the initial price of the non-standard option with pay-off

\[ Y = (\max(S(1), S(2), S(3)) - 30)_+ \]

and time of maturity \( T = 3 \) (max. 3 points). Compute the probability that
the derivative expires in the money (max. 1 point) and the probability that
the return of a constant portfolio with a long position on this derivative be
positive (max. 1 point).
Solution: (i) The binomial tree of the stock price is

\[ S(3) = 64 \]
\[ S(2) = 48 \]
\[ S(1) = 36 \]
\[ S(0) = 27 \]
\[ S(2) = 12 \]
\[ S(1) = 9 \]
\[ S(2) = 3 \]
\[ S(3) = 1 \]

To compute the initial price of a non-standard European derivative it is convenient to use the formula

\[ \Pi_Y(0) = e^{rN} \sum_{x \in \{u,d\}^N} q_x \cdot \ldots q_x Y(x), \]

where \( Y(x) \) denotes the pay-off as a function of the path of the stock price. In this problem we have \( N = 3, r = 0 \) and \( q_u = \frac{2}{3}, q_d = \frac{1}{3} \).

So, it remains to compute the pay-off for all possible paths of the stock price. We have

\[ Y(u, u, u) = 34, \quad Y(u, u, d) = 18, \quad Y(u, d, u) = 6 = Y(u, d, d). \]

The pay-offs for all other paths are zero. Thus,

\[ \Pi_Y(0) = \left( \frac{2}{3} \right)^3 \cdot 34 + \left( \frac{2}{3} \right)^2 \cdot \frac{1}{3} \cdot 18 + \left( \frac{2}{3} \right)^2 \cdot 3 \cdot 6 + \frac{2}{3} \cdot \left( \frac{1}{3} \right)^2 \cdot 6 = \frac{380}{27} \approx 14.1. \]
(ii) The probability that the derivative expires in the money is the probability that $Y > 0$. Hence we just sum the probabilities of the paths which lead to a positive pay-off:

$$
P(Y > 0) = P(\{u, u, u\}) + P(\{u, u, d\}) + P(\{u, d, u\}) + P(\{u, d, d\})
$$

$$
= p^3 + 2p^2(1-p) + p(1-p)^2 = p = \frac{7}{8}.
$$

(iii) Consider a constant portfolio with a long position on the derivative. This means that we buy the derivative at time $t = 0$ and we wait (without changing the portfolio) until the expiration time $t = 3$. The return will be positive if $\Pi_Y(3) > \Pi_Y(0)$. But $\Pi_Y(3) = Y$, which, according to the computations above, is smaller than $\Pi_Y(0) \approx 14.1$ when the stock price follows one of the paths $\{u, u, u\}, \{u, u, d\}$. Hence

$$
P[\Pi_Y(3) > \Pi_Y(0)] = p^3 + p^2(1-p) = p^2 = \frac{49}{64} \approx 76.6\%.
$$

2. (i) Define the arbitrage portfolio in a binomial market (max. 1 point);
(ii) Prove that the binomial market is arbitrage-free if and only if the market parameters are such that $d < r < u$ (max. 4 points).

Solution: Definition 2.4 and Theorem 2.3 in Lecture Notes.

3. Let $K > \Delta K > 0$. Consider a European style derivative on a stock with maturity $T > 0$ and pay-off $Y = g(S(T))$, where

$$
g(x) = (x - K + \Delta K)_+ - 2(x - K)_+ + (x - K - \Delta K)_+.
$$

Draw the pay-off function of the derivative (max. 1 point). Compute the Black-Scholes price of the derivative (max. 2 points). Compute the number of shares of stock in the self-financing portfolio hedging this derivative (max. 2 points).

Solution: (i) We have

$$
g(x) = \begin{cases} 
0, & x \leq K - \Delta K; \\
x - K + \Delta K, & K - \Delta K < x \leq K; \\
K + \Delta K - x, & K < x \leq K + \Delta K; \\
0, & x > K + \Delta K.
\end{cases}
$$

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The drawing is straightforward now.

(ii) We can write \( g(x) = g_1(x) - 2g_2(x) + g_3(x) \), where

\[
g_1(x) = (x - K + \Delta K)_+, \quad g_2(x) = (x - K)_+, \quad g_3(x) = (x - K - \Delta K)_+
\]

are the pay-off functions of European calls with strikes \( K - \Delta K \), \( K \) and \( K + \Delta K \), respectively. As the Black-Scholes price is linear in the pay-off function, the Black-Scholes price of the derivative is a linear combination of the Black-Scholes price of the derivatives with pay-off functions \( g_1, g_2, g_3 \).

Hence,

\[
\Pi_Y(t) = C(t, S(t), K - \Delta K, T) - 2C(t, S(t), K, T) + C(t, S(t), K + \Delta K, T).
\]

It follows from Theorem 6.5 in Lecture Notes that

\[
C(t, x, K, T) = x\Phi(d_1) - Ke^{-rT}\Phi(d_2),
\]

where

\[
d_2 = \frac{\log \left( \frac{x}{K} \right) + (r - \frac{1}{2}\sigma^2) \tau}{\sigma \sqrt{\tau}}, \quad d_1 = d_2 + \sigma \sqrt{\tau},
\]

and where \( \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}y^2} dy \) is the standard normal distribution.

Hence,

\[
C(t, x, K - \Delta K, T) = x\Phi(a_1) - (K - \Delta K)e^{-rT}\Phi(a_2),
\]

where

\[
a_2 = \frac{\log \left( \frac{x}{K - \Delta K} \right) + (r - \frac{1}{2}\sigma^2) \tau}{\sigma \sqrt{\tau}} = \frac{\log \left( \frac{x}{K} \right) + \log \left( \frac{K}{K - \Delta K} \right) + (r - \frac{1}{2}\sigma^2) \tau}{\sigma \sqrt{\tau}}
\]

\[
= d_2 + \frac{\log \left( \frac{K}{K - \Delta K} \right)}{\sigma \sqrt{\tau}},
\]

\[
a_1 = a_2 + \sigma \sqrt{\tau} = d_2 + \frac{\log \left( \frac{K}{K - \Delta K} \right)}{\sigma \sqrt{\tau}} + \sigma \sqrt{\tau}.
\]

Similarly,

\[
C(t, x, K + \Delta K, T) = x\Phi(b_1) - (K + \Delta K)e^{-rT}\Phi(b_2),
\]
where
\[ b_2 = d_2 + \log \left( \frac{K}{K + \Delta K} \right) \frac{1}{\sigma \sqrt{\tau}}, \]
\[ b_1 = d_2 + \log \left( \frac{K}{K + \Delta K} \right) + \sigma \sqrt{\tau}. \]

Denote
\[ \epsilon_+ := \frac{\log \left( \frac{K}{K + \Delta K} \right)}{\sigma \sqrt{\tau}} \]
\[ \epsilon_- := \frac{\log \left( \frac{K - \Delta K}{K - \Delta K} \right)}{\sigma \sqrt{\tau}}. \]

We obtain
\[ \Pi_Y(t) = x \Phi(d_1 + \epsilon_-) - (K - \Delta K)e^{-r\tau} \Phi(d_2 + \epsilon_-) \]
\[ - 2 \left[ x \Phi(d_1) - K e^{-r\tau} \Phi(d_2) \right] \]
\[ + x \Phi(d_1 + \epsilon_+) - (K + \Delta K)e^{-r\tau} \Phi(d_2 + \epsilon_+), \text{ at } x = S(t). \]

Rearranging, we obtain
\[ \Pi_Y(t) = x \left[ \Phi(d_1 + \epsilon_-) - 2 \Phi(d_1) + \Phi(d_1 + \epsilon_+) \right] \]
\[ - K e^{-r\tau} \left[ \Phi(d_2 + \epsilon_-) - 2 \Phi(d_2) + \Phi(d_2 + \epsilon_+) \right] \]
\[ + \Delta K e^{-r\tau} \left[ \Phi(d_2 + \epsilon_-) - \Phi(d_2 + \epsilon_+) \right], \text{ at } x = S(t). \]

(iii) The number of shares of stock in the self-financing portfolio hedging this derivative is given by
\[ h_S(t) = \frac{\partial}{\partial x} v(t, x)|_{x=S(t)}, \]

where \( v \) is such that \( \Pi_Y(t) = v(t, S(t)) \).

It is known from Theorem 6.6 in Lecture Notes that the number of shares of stock in the self-financing portfolio hedging the European call is \( \Phi(d_1) \).

Thus, in our case
\[ h_S(t) = \Phi(d_1 + \epsilon_-) - 2 \Phi(d_1) + \Phi(d_1 + \epsilon_+). \]