Exam for the course Options and Mathematics (CTH[MVE095], GU[MMA700]) 2017/18

Telefonvakt/Rond: Jonatan Kallus (5325)
January, 12 2018

REMARKS: No aids permitted

1. Consider a 3-period binomial asset pricing model with the following parameters:

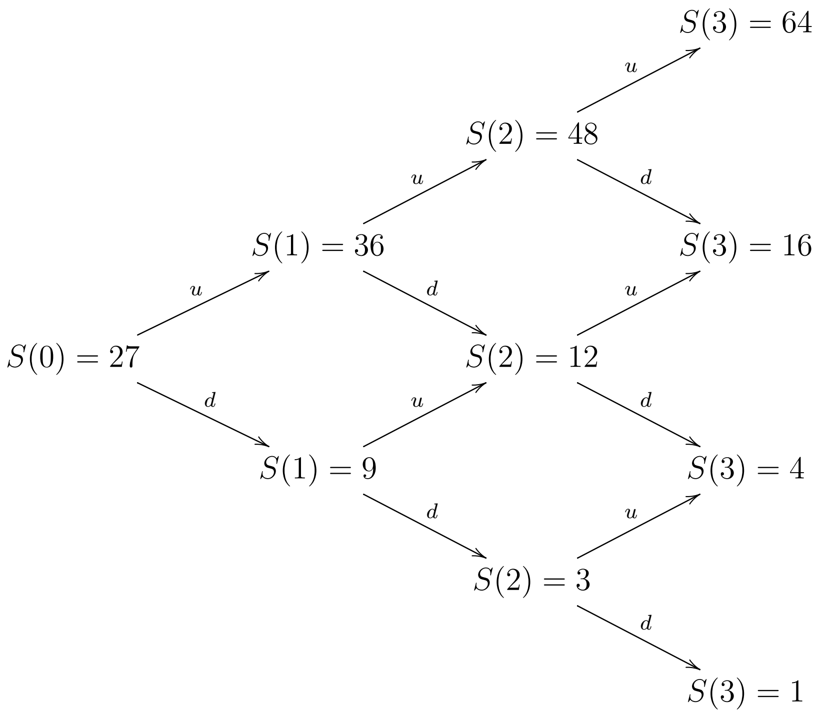
$$e^u = \frac{4}{3}, \quad e^d = \frac{1}{3}, \quad p = \frac{7}{8}.$$

Assuming $S(0) = 27$ and that the risk-free asset has zero interest rate ($r = 0$), compute the initial price of the non-standard option with pay-off

$$Y = (\max(S(1), S(2), S(3)) - 30)_+$$

and time of maturity $T = 3$ (max. 3 points). Compute the probability that the derivative expires in the money (max. 1 point) and the probability that the return of a constant portfolio with a long position on this derivative be positive (max. 1 point).

Solution: (i) The binomial tree of the stock price is



To compute the initial price of a non-standard European derivative it is convenient to use the formula

$$\Pi_Y(0) = e^{rN} \sum_{x \in \{u,d\}^N} q_{x_1} \cdot \ldots q_{x_N} Y(x),$$

where $Y(x)$ denotes the pay-off as a function of the path of the stock price. In this problem we have $N = 3$, $r = 0$ and $q_u = \frac{2}{3}$, $q_d = \frac{1}{3}$.

So, it remains to compute the pay-off for all possible paths of the stock price. We have

$$Y(u,u,u) = 34, \quad Y(u,u,d) = 18, \quad Y(u,d,u) = 6 = Y(u,d,d).$$

The pay-offs for all other paths are zero. Thus,

$$\Pi_Y(0) = \left(\frac{2}{3}\right)^3 \cdot 34 + \left(\frac{2}{3}\right)^2 \cdot \frac{1}{3} \cdot 18 + \left(\frac{2}{3}\right)^2 \cdot \frac{1}{3} \cdot 6 + \frac{2}{3} \cdot \left(\frac{1}{3}\right)^2 \cdot 6 = \frac{380}{27} \approx 14.1.$$

(ii) The probability that the derivative expires in the money is the probability that $Y > 0$. Hence we just sum the probabilities of the paths which lead to a positive pay-off:

$$\mathbb{P}(Y > 0) = \mathbb{P}(\{u, u, u\}) + \mathbb{P}(\{u, u, d\}) + \mathbb{P}(\{u, d, u\}) + \mathbb{P}(\{u, d, d\})$$

$$= p^3 + 2p^2(1-p) + p(1-p)^2 = p = \frac{7}{8}.$$

(iii) Consider a constant portfolio with a long position on the derivative. This means that we buy the derivative at time $t = 0$ and we wait (without changing the portfolio) until the expiration time $t = 3$. The return will be positive if $\Pi_Y(3) > \Pi_Y(0)$. But $\Pi_Y(3) = Y$, which, according to the computations above, is smaller than $\Pi_Y(0) \approx 14.1$ when the stock price follows one of the paths $\{u, u, u\}$, $\{u, u, d\}$. Hence

$$\mathbb{P}[\Pi_Y(3) > \Pi_Y(0)] = p^3 + p^2(1-p) = p^2 = \frac{49}{64} \approx 76.6\%.$$

2. (i) Define the arbitrage portfolio in a binomial market (max. 1 point);
(ii) Prove that the binomial market is arbitrage-free if and only if the market parameters are such that $d < r < u$ (max. 4 points).
Solution: Definition 2.4 and Theorem 2.3 in Lecture Notes.

3. Let $K > \Delta K > 0$. Consider a European style derivative on a stock with maturity $T > 0$ and pay-off $Y = g(S(T))$, where

$$g(x) = (x - K + \Delta K)_+ - 2(x - K)_+ + (x - K - \Delta K)_+.$$

Draw the pay-off function of the derivative (max. 1 point). Compute the Black-Scholes price of the derivative (max. 2 points). Compute the number of shares of stock in the self-financing portfolio hedging this derivative (max. 2 points).
Solution: (i) We have

$$g(x) = \begin{cases} 0, & x \leq K - \Delta K; \\ x - K + \Delta K, & K - \Delta K < x \leq K; \\ K + \Delta K - x, & K < x \leq K + \Delta K; \\ 0, & x > K + \Delta K. \end{cases}$$

The drawing is straightforward now.

(ii) We can write $g(x) = g_1(x) - 2g_2(x) + g_3(x)$, where

$$g_1(x) = (x - K + \Delta K)_+, \quad g_2(x) = (x - K)_+, \quad g_3(x) = (x - K - \Delta K)_+$$

are the pay-off functions of European calls with strikes $K - \Delta K$, K and $K + \Delta K$, respectively. As the Black-Scholes price is linear in the pay-off function, the Black-Scholes price of the derivative is a linear combination of the Black-Scholes price of the derivatives with pay-off functions g_1, g_2, g_3. Hence,

$$\Pi_Y(t) = C(t, S(t), K - \Delta K, T) - 2C(t, S(t), K, T) + C(t, S(t), K + \Delta K, T).$$

It follows from Theorem 6.5 in Lecture Notes that

$$C(t, x, K, T) = x\Phi(d_1) - Ke^{-r\tau}\Phi(d_2),$$

where

$$d_2 = \frac{\log\left(\frac{x}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}, \quad d_1 = d_2 + \sigma\sqrt{\tau},$$

and where $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}y^2}\,dy$ is the standard normal distribution.

Hence,

$$C(t, x, K - \Delta K, T) = x\Phi(a_1) - (K - \Delta K)e^{-r\tau}\Phi(a_2),$$

where

$$a_2 = \frac{\log\left(\frac{x}{K-\Delta K}\right) + \left(r - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}} = \frac{\log\left(\frac{x}{K}\right) + \log\left(\frac{K}{K-\Delta K}\right) + \left(r - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}$$

$$= d_2 + \frac{\log\left(\frac{K}{K-\Delta K}\right)}{\sigma\sqrt{\tau}},$$

$$a_1 = a_2 + \sigma\sqrt{\tau} = d_2 + \frac{\log\left(\frac{K}{K-\Delta K}\right)}{\sigma\sqrt{\tau}} + \sigma\sqrt{\tau}.$$

Similarly,

$$C(t, x, K + \Delta K, T) = x\Phi(b_1) - (K + \Delta K)e^{-r\tau}\Phi(b_2),$$

where

$$b_2 = d_2 + \frac{\log\left(\frac{K}{K+\Delta K}\right)}{\sigma\sqrt{\tau}},$$

$$b_1 = d_2 + \frac{\log\left(\frac{K}{K+\Delta K}\right)}{\sigma\sqrt{\tau}} + \sigma\sqrt{\tau}.$$

Denote

$$\epsilon_+ := \frac{\log\left(\frac{K}{K+\Delta K}\right)}{\sigma\sqrt{\tau}}, \quad \epsilon_- := \frac{\log\left(\frac{K}{K-\Delta K}\right)}{\sigma\sqrt{\tau}}.$$

We obtain

$$\begin{aligned}\Pi_Y(t) &= x\Phi(d_1+\epsilon_-) - (K-\Delta K)e^{-r\tau}\Phi(d_2+\epsilon_-) \\ &\quad - 2\left[x\Phi(d_1) - Ke^{-r\tau}\Phi(d_2)\right] \\ &\quad + x\Phi(d_1+\epsilon_+) - (K+\Delta K)e^{-r\tau}\Phi(d_2+\epsilon_+), \quad \text{at } x = S(t).\end{aligned}$$

Rearranging, we obtain

$$\begin{aligned}\Pi_Y(t) &= x\left[\Phi(d_1+\epsilon_-) - 2\Phi(d_1) + \Phi(d_1+\epsilon_+)\right] \\ &\quad - Ke^{-r\tau}\left[\Phi(d_2+\epsilon_-) - 2\Phi(d_2) + \Phi(d_2+\epsilon_+)\right] \\ &\quad + \Delta K e^{-r\tau}\left[\Phi(d_2+\epsilon_-) - \Phi(d_2+\epsilon_+)\right], \quad \text{at } x = S(t).\end{aligned}$$

(iii) The number of shares of stock in the self-financing portfolio hedging this derivative is given by

$$h_S(t) = \frac{\partial}{\partial x} v(t,x)|_{x=S(t)},$$

where v is such that $\Pi_Y(t) = v(t, S(t))$.

It is known from Theorem 6.6 in Lecture Notes that the number of shares of stock in the self-financing portfolio hedging the European call is $\Phi(d_1)$. Thus, in our case

$$h_S(t) = \Phi(d_1+\epsilon_-) - 2\Phi(d_1) + \Phi(d_1+\epsilon_+).$$