Exam for the course "Options and Mathematics" (CTH[*MVE095*], GU[*MMA700*]) 2016/17

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August 136 2017

REMARKS: (1) No aids permitted

1. (i) Define and explain the concept of arbitrage portfolio process invested in a binomial market (max. 1 point)

   (ii) Assume that the dominance principle holds. Prove the put-call parity (max. 2 points). Prove that the price of European call options is a convex function of the strike price (max. 2 points).

   Solution. See Definition 2.4, and Theorem 1.2.

2. Consider a 3-period binomial market with the following parameters:

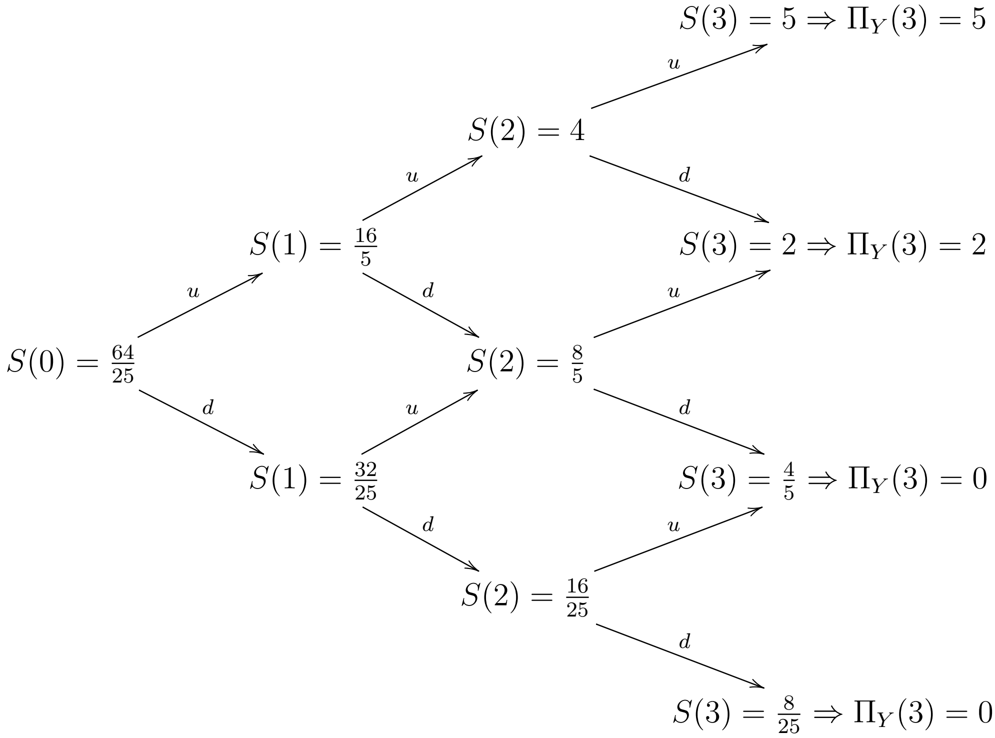
$$e^u = \frac{5}{4}, \quad e^d = \frac{1}{2}, \quad e^r = 1 \quad p = \frac{1}{2}.$$

Assume $S_0 = \frac{64}{25}$. Consider the European digital call option expiring at time $T = 3$ and with pay-off

$$Y = S(3)H(S(3) - 1),$$

where H is the Heaviside function: $H(x) = 0$, if $x < 0$, $H(x) = 1$ if $x \geq 0$. Compute the possible paths of the price $\Pi_Y(t)$ of the derivative (max. 1 point). Compute the expectation of $\Pi_Y(t)$ in the risk-neutral probability measure at each time $t \in \{0, 1, 2, 3\}$ and explain the obtained result (max. 2 points). Next consider a portfolio which is long x shares of the stock and short 1 share of the digital option. Show that if x is too large the expected return of this portfolio is negative (max. 2 points).

Solution. We start by writing down the diagram of the stock price and the value of the derivative at time of maturity $T = 3$ (which is equal to the pay-off)



The parameters of the binomial model are such that

$$q_u = \frac{2}{3}, \quad q_d = \frac{1}{3}, \quad r = 0.$$

To compute the price of the derivative at the times $t \in \{0, 1, 2\}$ we use the recurrence formula

$$\Pi_Y(t) = e^{-r}(q_u \Pi_Y^u(t+1) + q_d \Pi_Y^d(t+1)) = \frac{2}{3}\Pi_Y^u(t+1) + \frac{1}{3}\Pi_Y^d(t+1), \quad t \in \{0, 1, 2\}.$$

Hence at time $t = 2$ we have

$$\begin{aligned} S(2) = 4 &\Rightarrow \Pi_Y(2) = \frac{2}{3} \cdot 5 + \frac{1}{3} \cdot 2 = 4 \\ S(2) = \frac{8}{5} &\Rightarrow \Pi_Y(2) = \frac{2}{3} \cdot 2 + \frac{1}{3} \cdot 0 = \frac{4}{3} \\ S(2) = \frac{16}{25} &\Rightarrow \Pi_Y(2) = \frac{2}{3} \cdot 0 + \frac{1}{3} \cdot 0 = 0. \end{aligned}$$

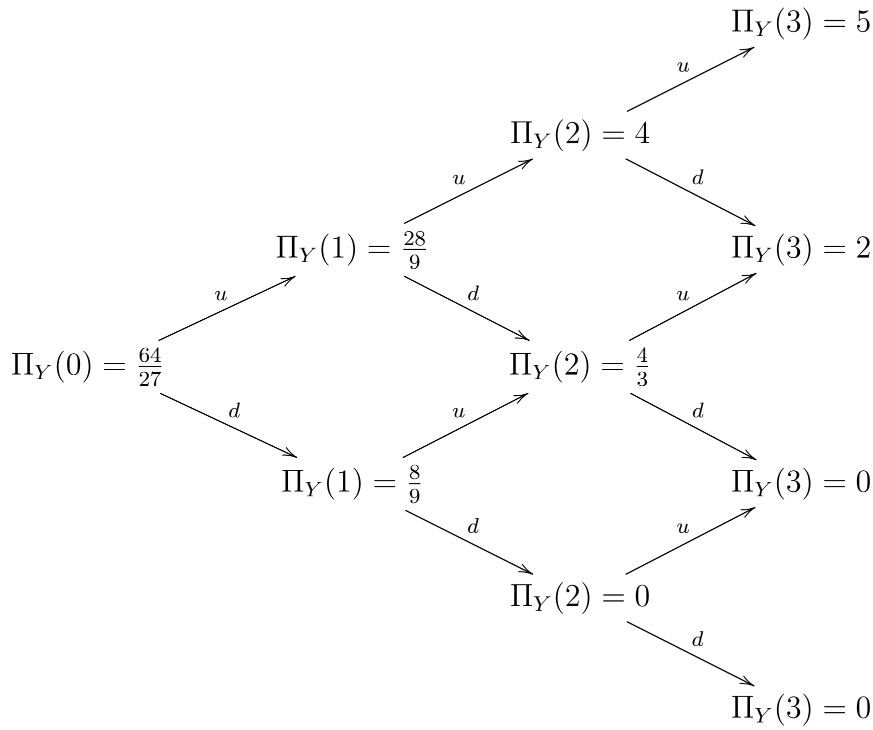
At time $t = 1$ we have

$$\begin{aligned} S(1) = \frac{16}{5} &\Rightarrow \Pi_Y(1) = \frac{2}{3} \cdot 4 + \frac{1}{3} \cdot \frac{4}{3} = \frac{28}{9} \\ S(1) = \frac{32}{25} &\Rightarrow \Pi_Y(1) = \frac{2}{3} \cdot \frac{4}{3} + \frac{1}{3} \cdot 0 = \frac{8}{9} \end{aligned}$$

and at time $t = 0$ we have

$$\Pi_Y(0) = \frac{2}{3} \cdot \frac{28}{9} + \frac{1}{3} \cdot \frac{8}{9} = \frac{64}{27}$$

Hence we obtain the following diagram for the derivative price



This concludes the first part of the exercise (1 point).

Let $\mathbb{E}_q[\Pi_Y(t)]$ denote the expectation of $\Pi_Y(t)$ at time t in the risk-neutral probability measure (q_u, q_d). Clearly $\mathbb{E}_q[\Pi_Y(0)] = \Pi_Y(0) = 64/27$. At time $t = 1$ we have

$$\mathbb{E}_q[\Pi_Y(1)] = q_u \frac{28}{9} + q_d \frac{8}{9} = \frac{2}{3}\frac{28}{9} + \frac{1}{3}\frac{8}{9} = \frac{64}{27}.$$

At time $t = 2$,

$$\mathbb{E}_q[\Pi_Y(2)] = q_u^2 \cdot 4 + 2 q_u q_d \frac{4}{3} + q_d^2 \cdot 0 = \left(\frac{2}{3}\right)^2 4 + 2\frac{2}{3}\frac{1}{3}\frac{4}{3} = \frac{64}{27}.$$

At time $t = 3$ we have

$$\mathbb{E}_q[\Pi_Y(3)] = \left(\frac{2}{3}\right)^3 5 + 3\left(\frac{2}{3}\right)^2 \frac{1}{3} 2 = \frac{64}{27}.$$

Hence the expectation of the derivative price in the risk neutral measure is time independent. This is explained by the fact that the risk-free rate is zero, hence the price of the derivative equals its discounted price. Since the latter is a martingale in the risk neutral probability measure, then it has constant expectation in this probability measure. This concludes the second part of the exercise (2 points).

The value at time t of a portfolio with x shares of the stock and -1 share of the derivative is

$$V(t) = xS(t) - \Pi_Y(t) \Rightarrow \mathbb{E}[R] = x(\mathbb{E}[S(T)] - S(0)) + \mathbb{E}[Y] - \Pi_Y(0) = ax + b.$$

Note that the expected return is computed with the physical probability p and *not* with the risk-neutral probability. As $\mathbb{E}[S(T)] = (1/2)^3(5 + 2 + 4/5 + 8/25) = 203/200 < 64/25 = S(0)$, then $a < 0$. Hence $\mathbb{E}[R] > 0$ if and only if $x < b/|a|$. This concludes the third part of the exercise.

3. Consider a European derivative with pay-off $Y = (S(T) - S(0))^2/S(T)$ at time of maturity $T > 0$. Compute the Black-Scholes price $\Pi_Y(t)$ (max. 2 points) and the number of shares of the stock in the hedging portfolio (max. 1 point). Compute the lowest possible value of $\Pi_Y(t)$ (max. 2 points).

   Solution. Since $Y = S(T) - 2S(0) + S(0)^2 S(T)^{-1} = Y_1 + Y_2 + Y_3$, and since the Black-Schole price $\Pi_Y(t)$ is linear in the pay-off, then

$$\Pi_Y(t) = \Pi_{Y_1}(t) + \Pi_{Y_2}(t) + \Pi_{Y_3}(t).$$

   Recall that, for $Y = g(S(T))$, the Black-Scholes price is $\Pi_Y(t) = v(t, S(t))$, where

$$v(t,x) = \frac{e^{-r\tau}}{\sqrt{2\pi}} \int_{\mathbb{R}} g(xe^{(r-\frac{\sigma^2}{2})\tau} e^{\sigma\sqrt{\tau}y}) e^{-\frac{y^2}{2}}\, dy.$$

   In this case we have $g_1(x) = x$, $g_2(x) = -2S(0)$ (constant) and $g_3(x) = S(0)^2/x$. Computing the resulting integral we obtain

$$\Pi_Y(t) = S(t) - 2S(0)e^{-r\tau} + S(0)^2 e^{(\sigma^2-2r)\tau} S(t)^{-1}.$$

   This completes the first part of the exercise (2 points). Note that $\Pi_Y(t) = v(t, S(t))$, where

$$v(t,x) = x - 2S(0)e^{-r\tau} + S(0)^2 e^{(\sigma^2-2r)\tau}/x.$$

   Hence

$$h_S(t) = \partial_x v(t, S(t)) = 1 - S(0)^2 e^{(\sigma^2-2r)\tau} S(t)^{-2}.$$

   This concludes the second part of the exercise (1 point). Now, the price function $v(t,x)$ has only one minimum at $x : \partial_x v = 0$, that is at the price $S(t)$ for which $h_S(t) = 0$. From the formula for $h_S(t)$, we see that

$$h_S(t) = 0 \quad \text{if and only if} \quad S(t) = S(0)e^{\frac{1}{2}(\sigma^2-2r)\tau}.$$

   Replacing this value of $S(t)$ in the formula for $\Pi_Y(t)$ we obtain that the minimal value of $\Pi_Y(t)$ is given by $2S(0)e^{-r\tau}(e^{\sigma^2/2} - 1)$. This concludes the third part of the exercise (2 points)