Exam for the course “Options and Mathematics”
(CTH[MVE095], GU[MMA700]) 2015/16

Telefonvakt/Rond: Mattias Lennartsson, TEL. 5325
August 17, 2016

REMARK: No aids permitted

1. Assume that the dominance principle holds and that there exists a risk-free asset with constant interest rate $r$. Prove the following:

- the put-call parity (max. 2 points)
- if $r \geq 0$, the price of call options is non-decreasing with the time of maturity (max. 1 point)
- the price of call options is convex in the strike price (max. 1 point)

Define and explain the concept of optimal exercise time of American put options (max. 1 point).

**Solution:** See Theorem 1.1 and Def. 1.1 in the lecture notes.

2. Let the price $S(t)$ of a stock be given by a $N$-period binomial model with parameters $u > 0$, $d < 0$, $0 < r < u$, $p \in (0,1)$ and let $\hat{\Pi}(t)$ be the binomial price of an American put on the stock with strike $K > 0$ and maturity $T = N$. Express $\hat{\Pi}(N-1)$ as a function of $S(N-1)$ (max. 2 points). Show that it is optimal to exercise the American put at time $t = N - 1$ if and only if the price of the stock at this time satisfies

$$S(N - 1) \leq K \frac{1 - e^{-r}q_d}{1 - e^{-r}q_de^d} \quad (\text{max. 3 points}).$$

**Solution:** By definition of binomial price of American put options we have

$$\hat{\Pi}(N) = (K - S(N))_+, \quad \hat{\Pi}(N - 1) = \max\{(K - S(N - 1))_+, e^{-r}(q_u\hat{\Pi}^u(N) + q_d\hat{\Pi}^d(N))\}$$

Using that

$$\hat{\Pi}^u(N) = (K - S(N - 1)e_u)_+, \quad \hat{\Pi}^d(N) = (K - S(N - 1)e_d)_+,$$
we obtain \( \hat{\Pi}(N-1) = f(S(N-1)) \), where

\[
f(x) = \max\{ (K-x)_+, e^{-r}(q_u e^u (K e^{-u} - x)_+ + q_d e^d (K e^{-d} - x)_+) \}.\]

This concludes the first part of the exercise (2 points). For the second part of the exercise, we recall that it is optimal to exercise the derivative at time \( t = N - 1 \) if and only if \( \hat{\Pi}(N-1) = (K - S(N-1))_+ \), i.e., if and only if the binomial price of the American put equals its intrinsic value. To see when this happens, we compute \( \hat{\Pi}(N-1) \) when \( S(N-1) \) lies in the intervals

\[
S(N-1) \in [0, K e^{-u}] := I_1, \quad S(N-1) \in [K e^{-u}, K] := I_2, \\
S(N-1) \in [K, K e^{-d}] := I_3, \quad S(N-1) \in [K e^{-d}, +\infty) := I_4
\]

Using the formula \( \hat{\Pi}(N-1) = f(S(N-1)) \) proved above, we see that, for \( S(N-1) \in I_1 \),

\[
\hat{\Pi}(N-1) = \max\{K - S(N-1), e^{-r}(q_u e^u (K e^{-u} - S(N-1)) + q_d e^d (K e^{-d} - S(N-1)))\}.
\]

Using \( q_u + q_d = 1 \) and \( q_u e^u + q_d e^d = e^r \) we obtain

\[
\hat{\Pi}(N-1) = \max\{K - S(N-1), Ke^{-r} - S(N-1)\} = K - S(N-1), \quad \text{for } S(N-1) \in I_1.
\]

Similarly, for \( S(N-1) \in I_2 \) we have

\[
\hat{\Pi}(N-1) = \max\{K - S(N-1), e^{-r} q_d e^d (K e^{-d} - S(N-1))\}
= \begin{cases} 
K - S(N-1) & \text{for } S(N-1) \leq S_* \\
- e^{-r} q_d e^d (K e^{-d} - S(N-1)) & \text{for } S(N-1) > S_* 
\end{cases}
\]

where

\[
S_* = K \frac{1 - e^{-r} q_d}{1 - e^{-r} q_d e^d}.
\]

Treating similarly the cases \( S(N-1) \in I_3 \) and \( S(N-1) \in I_4 \) we find

\[
\hat{\Pi}(N-1) = \begin{cases} 
K - S(N-1) & \text{for } 0 < S(N-1) \leq S_* \\
e^{-r} q_d e^d (K e^{-d} - S(N-1)) & \text{for } S_* < S(N-1) \leq K e^{-d} \\
0 & \text{for } S > K e^{-d}
\end{cases}
\]

We conclude that it is optimal to exercise the American put at time \( t = N - 1 \) if and only if \( S(N-1) \leq S_* \), which completes the solution of the second part of the exercise (3 points).

3. Let \( 0 < L < K \). A European style derivative on a stock with maturity \( T > 0 \) pays nothing to its owner when \( S(T) > K \), while for \( S(T) \leq K \) it lets the owner choose between 1 share of the stock and the fixed amount \( L \). Draw the pay-off function of the derivative (max. 1 point). Compute the Black-Scholes price of the derivative (max.
2 points). Compute the number of shares of the stock in the hedging self-financing portfolio (max. 2 points).

**Solution:** The pay-off function is

\[
g(z) = \begin{cases} 
L, & \text{for } 0 \leq z \leq L \\
z, & \text{for } L \leq z \leq K \\
0, & \text{for } z > K,
\end{cases}
\]

which is depicted in the figure.

The Black-Scholes price of the derivative is given by

\[
\Pi(t) = v(t, S(t)),
\]

where \(v(t, x) = \frac{e^{-r\tau}}{\sqrt{2\pi}} \int_{-\infty}^{d_1(L)} xe^{(r-\frac{\sigma^2}{2})\tau} e^{\sigma\sqrt{\tau}y} e^{-\frac{y^2}{2}} dy + \frac{e^{-r\tau}}{\sqrt{2\pi}} \int_{d_1(L)}^{d_1(K)} xe^{(r-\frac{\sigma^2}{2})\tau} e^{\sigma\sqrt{\tau}y} e^{-\frac{y^2}{2}} dy + \frac{e^{-r\tau}}{\sqrt{2\pi}} \int_{d_1(K)}^{\infty} xe^{(r-\frac{\sigma^2}{2})\tau} e^{\sigma\sqrt{\tau}y} e^{-\frac{y^2}{2}} dy.
\]

Replacing the pay-off function above we find:

\[
v(t, x) = \frac{e^{-r\tau}}{\sqrt{2\pi}} \int_{-\infty}^{d_1(L)} xe^{(r-\frac{\sigma^2}{2})\tau} e^{\sigma\sqrt{\tau}y} e^{-\frac{y^2}{2}} dy + \frac{e^{-r\tau}}{\sqrt{2\pi}} \int_{d_1(L)}^{d_1(K)} xe^{(r-\frac{\sigma^2}{2})\tau} e^{\sigma\sqrt{\tau}y} e^{-\frac{y^2}{2}} dy + \frac{e^{-r\tau}}{\sqrt{2\pi}} \int_{d_1(K)}^{\infty} xe^{(r-\frac{\sigma^2}{2})\tau} e^{\sigma\sqrt{\tau}y} e^{-\frac{y^2}{2}} dy
\]

\[
= Le^{-r\tau}\Phi(d_1(L)) + x[\Phi(d_2(K)) - \Phi(d_2(L))],
\]

where \(\Phi(z)\) is the standard normal distribution, \(d_2(a) = d_1(a) - \sigma\sqrt{\tau}\) and

\[
d_1(a) = \log \frac{L}{x} - \frac{r - \frac{\sigma^2}{2}}{\sigma\sqrt{\tau}}.
\]

This concludes the second part of the exercise (2 points). The number of shares of the stock in the hedging self-financing portfolio is 

\[
h_S(t) = \partial_x v(t, S(t)).
\]

We use

\[
\partial_x[\Phi(d_1(a))] = \phi(d_1(a))\partial_x[d_1(a)] = -\frac{\phi(d_1(a))}{x\sigma\sqrt{\tau}},
\]

where \(\phi(x)\) is the probability density function of the standard normal distribution.
\[ \partial_x[\Phi(d_2(a))] = -\frac{\phi(d_2(a))}{x\sigma\sqrt{\tau}} \]

where \( \phi(z) = \frac{e^{-z^2/2}}{\sqrt{2\pi}} \) is the standard normal density function. Hence

\[ \partial_x v(t, x) = -\frac{\phi(d_1(L))}{x\sigma\sqrt{\tau}} L e^{-r\tau} + \Phi(d_2(K)) - \Phi(d_2(L)) + \frac{\phi(d_2(L)) - \phi(d_2(K))}{\sigma\sqrt{\tau}} \]

The result can be further simplified by noticing that

\[ \phi(d_2(L)) - \phi(d_1(L)) \frac{L}{x} e^{-r\tau} = 0 \]

(see also sec. 6.2 in the lecture notes). Hence we finally obtain

\[ \partial_x v(t, x) = \Phi(d_2(K)) - \Phi(d_2(L)) - \frac{\phi(d_2(K))}{\sigma\sqrt{\tau}}. \]