Exam for the course “Options and Mathematics” (CTH[*MVE095*], GU[*MMA700*]) 2015/16

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REMARK: No aids permitted

1. Let $(\Omega, \mathbb{P})$ be a finite probability space and let $X = \{X_1, X_2, \dots\}$ be a discrete stochastic process.

   (a) Define and explain what it means that X is a martingale (max. 1 point)

   (b) Let $(\Omega_N, \mathbb{P}_p)$ be a N-coin toss probability space and consider a binomial market with $r \in (d, u)$. Show that there exists a unique probability measure $\mathbb{P}_q$ such that the discounted value of the stock is a martingale (max. 2 points)

   (b) Compute the expectation of the binomial stock price in the martingale probability measure (max. 2 points)

   Solution: See Definition 5.15 and Theorems 5.3-5.4 in the lecture notes.

2. Consider a 3-period binomial market with the following parameters:

$$e^u = \frac{5}{4}, \quad e^d = \frac{1}{2}, \quad e^r = 1, \quad S_0 = \frac{64}{25}, \quad p \in (0,1)$$

and a European derivative with maturity $T = 3$ and pay-off

$$Y = \Big(\frac{1}{4}\sum_{i=0}^{3} S(i) - 2\Big)_+,$$

which is an example of Asian call option. Compute the price of the derivative at time $t = 0$ (max. 3 points). Compute the probability that the derivative expires in the money (max. 1 point) and the probability that a long position in one share of the derivative gives a positive return (max. 1 point).

Solution: The derivative is non-standard, hence to compute $\Pi_Y(0)$ it is convenient to use directly its definition, namely

$$\Pi_Y(0) = e^{-3r} \sum_{x \in \{u,d\}^3} q_{x_1} q_{x_2} q_{x_3} Y(x)$$

where $Y(x) = \left(\frac{1}{4}\sum_{i=0}^{3} S(i,x) - 2\right)_+$. As $r = 0$ we have

$$\begin{aligned}\Pi_Y(0) &= (q_u)^3 Y(u,u,u) + (q_u)^2 q_d Y(u,u,d) + (q_u)^2 q_d Y(u,d,u) + q_u (q_d)^2 Y(u,d,d) \\ &\quad + (q_u)^2 q_d Y(d,u,u) + q_u (q_d)^2 Y(d,u,d) + q_u (q_d)^2 Y(d,d,u) + (q_d)^3 Y(d,d,d),\end{aligned}$$

where $q_u = \frac{e^r - e^d}{e^u - e^d} = \frac{2}{3}$ and $q_d = 1 - q_u = \frac{1}{3}$. By straightforward calculations,

$$Y(u,u,u) = \frac{169}{100}, \quad Y(u,u,d) = \frac{47}{50}, \quad Y(u,d,u) = \frac{17}{50}, \quad Y(u,d,d) = \frac{1}{25},$$

while the pay-off is zero along any other paths. Hence

$$\Pi_Y(0) = \left(\frac{2}{3}\right)^2 \frac{169}{100} + \left(\frac{2}{3}\right)^2 \frac{1}{3}\frac{47}{50} + \left(\frac{2}{3}\right)^2 \frac{1}{3}\frac{17}{50} + \frac{2}{3}\left(\frac{1}{3}\right)^2 \frac{1}{25} = \frac{52}{75}.$$

This concludes the first part of the exercise (3 points). The probability that the derivative expires in the money is

$$\begin{aligned}\mathbb{P}(Y > 0) &= \mathbb{P}(\{u,u,u\}) + \mathbb{P}(\{u,u,d\}) + \mathbb{P}(\{u,d,u\}) + \mathbb{P}(\{u,d,d\}) \\ &= p^3 + 2p^2(1-p) + p(1-p)^2 = p(p + (1-p))^2 = p.\end{aligned}$$

(in fact, the derivative expires in the money if and only if the stock price goes up at time 1). This concludes the second part of the exercise (1 point). The probability of a positive return is

$$\mathbb{P}(Y > \Pi_Y(0)) = \mathbb{P}(Y > \frac{52}{75}) = \mathbb{P}(\{u,u,u\}) + \mathbb{P}(\{u,u,d\}) = p^3 + p^2(1-p) = p^2$$

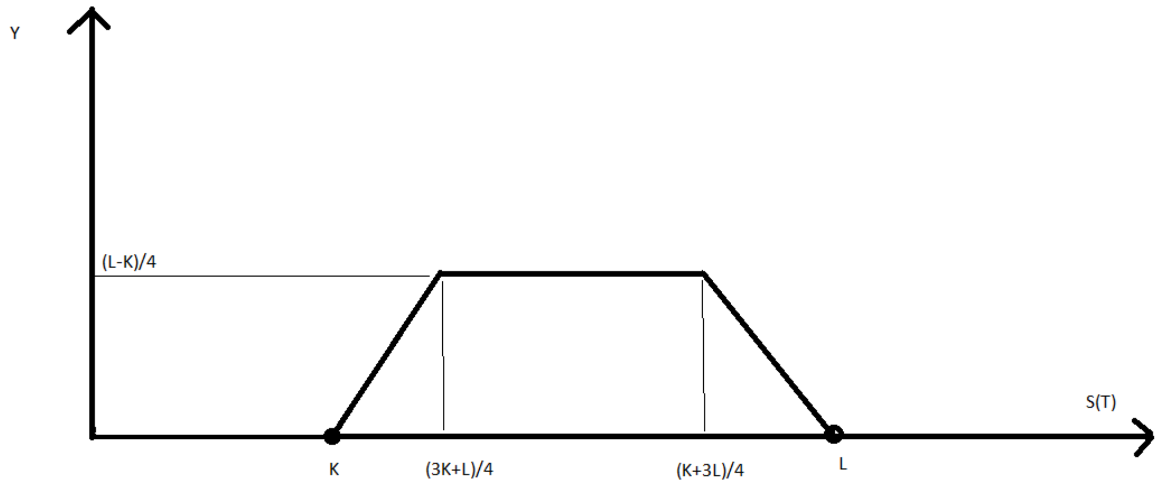
This concludes the third part of the exercise (1 point).

3. Let $0 < K < L$ and consider a European derivative with maturity $T > 0$ and pay-off

$$Y = \min[(S(T) - K)_+, (L - S(T))_+, \frac{1}{4}(L - K)].$$

Use the dominance principle to find a constant portfolio of European call options which replicates the derivative (max. 2 points). Compute the Black-Scholes price of the derivative (max. 2 points) and, assuming that the underlying stock follows a geometric Brownian motion, compute the probability that the derivative expires in the money (max. 1 point). Express the result of the last two computations in terms of the standard normal distribution.

Solution: Sketching the pay-off function on the plane we obtain the following picture.



By this one sees that

$$Y = \Big(S(T) - K\Big)_+ - \Big(S(T) - \frac{1}{4}(3K + L)\Big)_+ - \Big(S(T) - \frac{1}{4}(K + 3L)\Big)_+ + \Big(S(T) - L\Big)_+$$

By the dominance principle it follows that

$$\Pi_Y(t) = C(t, S(t), K, T) - C(t, S(t), \frac{1}{4}(3K+L), T) - C(t, S(t), \frac{1}{4}(K+3L), T) + C(t, S(t), L, T), \quad (1)$$

where $C(t, S(t), K, T)$ is the price of the call option with strike K. This concludes the first part of the exercise (2 points). To compute the Black-Scholes price we use the formula $\Pi_Y(t) = v(t, S(t))$, where

$$v(t,x) = e^{-r\tau}\int_{\mathbb{R}} g(xe^{(r-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}y})e^{-\frac{y^2}{2}}\frac{dy}{\sqrt{2\pi}}, \quad \tau = T - t,$$

where the pay-off function is

$$g(z) = \begin{cases} 0 & 0 < z \leq K \text{ or } z \geq L \\ z - K & K < z \leq \frac{3K+L}{4} \\ \frac{L-K}{4} & \frac{3K+L}{4} < z \leq \frac{K+3L}{4} \\ L - z & \frac{K+3L}{4} < z < L \end{cases}$$

Letting

$$d_1(a) = \frac{\log\frac{a}{x} - (r - \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}, \quad d_2(a) = d_1(a) - \sigma\sqrt{\tau}$$

we obtain

$$v(t,x) = e^{-r\tau}(I_1 + I_2 + I_3),$$

where

$$\begin{aligned} I_1 &= \int_{d_1(\frac{3K+L}{4})}^{d_1(K)} (xe^{(r-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}y} - K)e^{-\frac{y^2}{2}} \frac{dy}{\sqrt{2\pi}} \\ &= xe^{r\tau}[\Phi(d_2(K)) - \Phi(d_2(\tfrac{3K+L}{4})] - K[\Phi(d_1(K) - d_1(\tfrac{3K+L}{4})] \end{aligned}$$

$$\begin{aligned} I_2 &= \int_{d_1(\frac{3K+L}{4})}^{d_1(\frac{K+3L}{4})} \frac{L-K}{4} e^{-\frac{y^2}{2}} \frac{dy}{\sqrt{2\pi}} \\ &= \frac{L-K}{4}[\Phi(d_1(\tfrac{K+3L}{4}) - \Phi(d_1(\tfrac{3K+L}{4}))] \end{aligned}$$

$$\begin{aligned} I_3 &= \int_{d_1(\frac{K+3L}{4})}^{d_1(L)} (L - xe^{(r-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}y})e^{-\frac{y^2}{2}} \frac{dy}{\sqrt{2\pi}} \\ &= L[\Phi(d_1(L)) - \Phi(d_1(\tfrac{K+3L}{4}))] - xe^{r\tau}[\Phi(d_2(L)) - \Phi(d_2(\tfrac{K+3L}{4}))]. \end{aligned}$$

The same result can be obtained using the Black-Scholes formula for call options and the identity (1). This concludes the second part of the exercise (2 points).

Finally the probability that the derivative expires in the money is the probability that the stock price at maturity lies in the interval $[K, L]$, i.e.,

$$\mathbb{P}(Y > 0) = \mathbb{P}(K < S(T) < L).$$

Using $S(T) = S_0 e^{\alpha T + \sigma W(T)}$ we have

$$\mathbb{P}(Y > 0) = \mathbb{P}\Big(\frac{(\log(K/S_0) - \alpha T)}{\sigma\sqrt{T}} < \frac{W(T)}{\sqrt{T}} < \frac{(\log(L/S_0) - \alpha T)}{\sigma\sqrt{T}}\Big)$$

As $W(T)/\sqrt{T} \in \mathcal{N}(0,1)$ we obtain

$$\mathbb{P}(Y > 0) = \Phi(d(L)) - \Phi(d(K)),$$

where $d(z) = \frac{(\log(z/S_0) - \alpha T)}{\sigma\sqrt{T}}$. This concludes the third part of the exercise (1 point).