Exam for the course “Options and Mathematics” (CTH[MVE095], GU[MMA700]). Period 4, 2013/14

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REMARK: No aids permitted

1. Assume that the stock price $S(t)$ follows a 1-period binomial model with parameters $u > d$ and that the interest rate of the bond is $r > 0$. Show that there exists no self-financing arbitrage portfolio invested in the stock and the bond in the interval $t \in [0,1]$ if and only if $d < r < u$ (max 3 points). Show that any derivative on the stock expiring at time $t = 1$ can be hedged in this market (max 2 points).

Solution: See Theorem 3.2 (step 1) in Ref. [3] and Theorem 3.1 in Ref. [4]

2. Let $c(t)$ denote the Black-Scholes price at time $t$ of a European call with strike $K > 0$ and maturity $T > 0$ on a stock with price $S(t)$ and volatility $\sigma > 0$. Let $r > 0$ denote the interest rate of the bond. Compute the following limits:

$$\lim_{K \to 0^+} c(t), \quad \lim_{K \to +\infty} c(t), \quad \lim_{T \to +\infty} c(t), \quad \lim_{\sigma \to 0^+} c(t), \quad \lim_{\sigma \to +\infty} c(t).$$

Each limit gives 1 point if it is correct, 0 otherwise.

Solution: Recall that

$$c(t, x) = x \Phi(d_1) - Ke^{-rT} \Phi(d_2),$$

where

$$d_2 = \log \left( \frac{x}{K} \right) + \frac{r - \frac{1}{2}\sigma^2}{\sigma \sqrt{T}} t, \quad d_1 = d_2 + \sigma \sqrt{T},$$

and where $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}y^2} dy$ is the standard normal distribution. As $\sigma \to 0^+$ we have $d_1 \to d_2$ and

$$d_2 \sim \frac{1}{\sqrt{T}} \log \left( \frac{x}{K} + rT \right) \sigma^{-1}.$$

Hence

$$d_2 \to +\infty, \quad \text{if} \quad x > Ke^{-rT},$$

$$d_2 \to -\infty, \quad \text{if} \quad x < Ke^{-rT},$$

$$d_2 \to 0, \quad \text{if} \quad x = Ke^{-rT},$$

$$d_2 \to +\infty, \quad \text{if} \quad x > Ke^{-rT}.$$
Thus
\[
\lim_{\sigma \to 0^+} \Phi(d_1) = \lim_{\sigma \to 0^+} \Phi(d_2) = 1, \quad \text{if } x > Ke^{-rT}, \\
\lim_{\sigma \to 0^+} \Phi(d_1) = \lim_{\sigma \to 0^+} \Phi(d_2) = 0, \quad \text{if } x < Ke^{-rT}, \\
\lim_{\sigma \to 0^+} \Phi(d_1) = \lim_{\sigma \to 0^+} \Phi(d_2) = \Phi(0), \quad \text{if } x = Ke^{-rT}.
\]
It follows that
\[
\lim_{\sigma \to 0^+} c(t, x) = x - Ke^{-rT} \quad \text{if } x > Ke^{-rT}, \\
\lim_{\sigma \to 0^+} c(t, x) = 0, \quad \text{if } x \leq Ke^{-rT},
\]
i.e., \( \lim_{\sigma \to 0^+} c(t, x) = (x - Ke^{-rT})_+ \). For \( \sigma \to +\infty \) we have \( d_2 \to -\infty \) and \( d_1 \to +\infty \), hence \( \Phi(d_1) \to 1 \) and \( \Phi(d_2) \to 0 \). Thus \( c(t, x) \to x \) as \( \sigma \to +\infty \). As \( K \to 0^+ \), both \( d_1 \) and \( d_2 \) diverge to \( +\infty \), hence
\[
\lim_{K \to 0^+} c(t, x) = x.
\]
For \( K \to +\infty \), \( d_1, d_2 \) diverge to \( -\infty \). Hence the first term in \( c(t, x) \) converges to zero. As the first term in \( c(t, x) \) always dominates the second term (since \( c(t, x) > 0 \)), then the second term also goes to zero and thus
\[
\lim_{K \to +\infty} c(t, x) = 0.
\]
For \( T \to +\infty \) we have \( d_2 \to -\infty \) and \( d_1 \to +\infty \), hence
\[
\lim_{T \to +\infty} c(t, x) = x.
\]

3. Consider an American put option with strike \( K = 3/4 \) at the maturity time \( T = 2 \). Let the price \( S(t) \) of the underlying stock be given by the binomial model with parameters
\[
e^u = \frac{7}{4}, \quad e^d = \frac{1}{2}, \quad e^r = \frac{9}{8}.
\]
Assume \( S(0)=1 \). Compute the fair price of the derivative (max 2 points) and the hedging portfolio (max 2 points) at each time \( t = 0, 1, 2 \). Verify if the put-call parity holds at all times (max 1 point).
Solution: The binomial tree for the stock price is

When the price of the stock in the paths above is within a box, the put option is in the money. In fact, the binomial tree for the intrinsic value $Y(t)$ of the American put is

Now we compute the value $\hat{\Pi}_{put}(t)$ of the American put option. At time of maturity is given by the pay-off. At times $t = 0, 1$ we use the recurrence formula

$$\hat{\Pi}_{put}(t) = \max(Y(t), e^{-r}(q_u \hat{\Pi}_{put}^u(t + 1)) + q_d \hat{\Pi}_{put}^d(t + 1)),$$
where in this case we have \( q_u = q_d = 1/2 \). At time \( t = 1 \) we have
\[
\hat{\Pi}_{\text{put}}(1) = \max \left[ Y(1), \frac{4}{9}(\hat{\Pi}_{\text{put}}^u(2) + \hat{\Pi}_{\text{put}}^d(2)) \right] \\
= \max \left[ Y(1), \frac{4}{9} \left( \left( \frac{3}{4} - \frac{7}{4} S(1) \right)_+ + \left( \frac{3}{4} - \frac{1}{2} S(1) \right)_+ \right) \right].
\]
Since
\[
Y^u(1) = \left( \frac{3}{4} - \frac{7}{4} \right)_+ = 0, \quad Y^d(1) = \left( \frac{3}{4} - \frac{1}{2} \right)_+ = \frac{1}{4},
\]
we find
\[
\hat{\Pi}_{\text{put}}^u(1) = \max[0, 0] = 0, \quad \hat{\Pi}_{\text{put}}^d(1) = \max \left[ \frac{1}{4}, \frac{2}{9} \right] = \frac{1}{4}
\]
and so
\[
\hat{\Pi}_{\text{put}}(0) = \max \left[ Y(0), \frac{4}{9}(\hat{\Pi}_{\text{put}}^u(1) + \hat{\Pi}_{\text{put}}^d(1)) \right] = \frac{1}{9}.
\]
Hence the price of the American put corresponding to the different paths of the stock price is as follows:

This concludes the first part of the exercise (2 points). The hedging portfolio is computed by the formulas, for \( t = 1, 2 \),
\[
\hat{h}_S(t) = \frac{1}{S(t-1)} \frac{\hat{\Pi}_{\text{put}}^u(t) - \hat{\Pi}_{\text{put}}^d(t)}{e^u - e^d}, \tag{3}
\]
\[
\hat{h}_B(t) = \frac{e^{-r}}{B(t-1)} \frac{e^u \hat{\Pi}_{\text{put}}^d(t) - e^d \hat{\Pi}_{\text{put}}^u(t)}{e^u - e^d}. \tag{4}
\]
Hence
\[
\begin{align*}
    h_S(2) &= 0 \quad \text{if } S(1) = 7/4 \\
    h_S(2) &= -\frac{4}{5} \quad \text{if } S(1) = 1/2 \\
    h_S(1) &= -\frac{1}{5}.
\end{align*}
\]
\[
\begin{align*}
    h_B(2) &= 0 \quad \text{if } S(1) = 7/4 \\
    h_B(2) &= \frac{224}{405} \quad \text{if } S(1) = 1/2 \\
    h_B(1) &= \frac{14}{45} \frac{1}{B_0}.
\end{align*}
\]

where \( B_0 = B(0) \) is the initial value of the bond. This concludes the second part of the exercise (2 points). The put-call carity should not hold in this case, because the option is American. To verify this we compute first the fair price \( \hat{\Pi}_{call}(t) \) of the American call with the same parameters of the put option; we find easily

\[
\begin{align*}
    \hat{\Pi}_{call}(2) &= \frac{37}{16} \\
    \hat{\Pi}_{call}(1) &= \frac{39}{36} \\
    \hat{\Pi}_{call}(0) &= \frac{41}{81} \\
    \hat{\Pi}_{call}(2) &= \frac{1}{8} \\
    \hat{\Pi}_{call}(1) &= \frac{1}{18} \\
    \hat{\Pi}_{call}(2) &= 0
\end{align*}
\]

Letting \( Q(t) = \hat{\Pi}_{call}(t) - \hat{\Pi}_{put}(t) - S(t) + Ke^{-r(2-t)} \), \( t = 0, 1, 2 \), we find easily that \( Q = 0 \) only at maturity and when \( S(1) = 7/4 \).