REMARC: No aids permitted

1. Consider a 3-period binomial asset pricing model with the following parameters:

\[ e^u = \frac{4}{3}, \quad e^d = \frac{2}{3}, \quad p_u = \frac{3}{4}. \]

Assuming \( S(0) = 27 \) and that the bond has zero interest rate \( (r = 0) \), compute the initial price of the Lookback Option with pay-off

\[ Y = S(3) - \min(S(0), S(1), S(2), S(3)) \]

and time of maturity \( T = 3 \) (max. 3 points). Compute the probability that the derivative expires in the money (max. 1 point) and the probability that the return of a constant portfolio with a short position on this derivative be positive (max. 1 point).

**Solution:** The binomial tree of the stock price is

\[ \begin{align*}
S(3) &= 64 \\
S(2) &= 48 \\
S(1) &= 36 \\
S(0) &= 27
\end{align*} \]

\[ \begin{align*}
S(3) &= 32 \\
S(1) &= 24 \\
S(0) &= 18
\end{align*} \]

\[ \begin{align*}
S(3) &= 16 \\
S(2) &= 12 \\
S(0) &= 8
\end{align*} \]
To compute the initial price of a contingent claim it is convenient to use the formula

$$\Pi_Y(0) = e^{-rN} \sum_{x \in \{u,d\}^N} q_{x_1} \cdots q_{x_N} Y(x), \quad (1)$$

where $Y(x)$ denotes the pay-off as a function of the path of the stock price. In this exercise we have $N = 3, \; r = 0$ and

$$q_u = q_d = \frac{1}{2}.$$ 

So, it remains to compute the pay-off for all possible paths of the stock price, where

$$Y = S(3) - \min(S(0), S(1), S(2), S(3)).$$

For instance

$$Y(u, u, u) = 64 - \min(27, 36, 48, 64) = 64 - 27 = 37.$$ 

Similarly we find

$$Y(u, u, d) = 5, \quad Y(u, d, u) = 8, \quad Y(u, d, d) = 0$$

$$Y(d, u, u) = 14, \quad Y(d, u, d) = 0, \quad Y(d, d, u) = 4, \quad Y(d, d, d) = 0.$$ 

Replacing in (1) we obtain

$$\Pi_Y(0) = (q_u)^3 Y(u, u, u) + (q_u)^2 q_d Y(u, u, d) + (q_u)^2 q_d Y(u, d, u) + (q_d)^2 q_u Y(d, d, u)$$

$$+ (q_d)^2 q_u Y(d, u, u) + (q_d)^2 q_u Y(d, d, d) = \left( \frac{1}{2} \right)^3 (37 + 5 + 8 + 14 + 4) = \frac{17}{2} = 8.5.$$ 

This concludes the first part of the exercise (3 points). The probability that the derivative expires in the money is the probability that $Y > 0$. Hence we just sum the probabilities of the paths which lead to a positive pay-off:

$$\mathbb{P}(Y > 0) = \mathbb{P}\{\{u, u, u\}\} + \mathbb{P}\{\{u, u, d\}\} + \mathbb{P}\{\{u, d, u\}\} + \mathbb{P}\{\{d, u, u\}\} + \mathbb{P}\{\{d, d, u\}\}$$

$$= (p_u)^3 + 3(p_u)^2 p_d + (p_u)^2 p_d + (p_u)^3 + (p_d)^3 = \frac{27}{64} + \frac{3}{16} \frac{1}{4} + \frac{1}{16} \frac{3}{4} = \frac{57}{64} \approx 89\%.$$ 

This concludes the second part of the exercise (1 point). Next consider a constant portfolio with a short position on the derivative. This means that we buy the derivative at time $t = 0$ and we wait (without changing the portfolio) until the expiration time $t = 3$. The return will be positive if $\Pi_Y(3) < \Pi_Y(0)$. But $\Pi_Y(3) = Y$, which, according to the computations above, is smaller than $\Pi_Y(0) = 8.5$ when the stock price follows one of the paths $\{u, u, d\}, \{u, d, u\}, \{u, d, d\}, \{d, u, d\}, \{d, d, d\}$ or $\{d, d, u\}$. Hence

$$\mathbb{P}[\Pi_Y(3) < \Pi_Y(0)] = 2(p_u)^2 p_d + 3(p_d)^2 p_u + (p_d)^3 = \frac{7}{16} \approx 44\%.$$ 

This concludes the third part of the exercise (1 point).
2. Compute the Black-Scholes price of European calls and puts (max. 3 points). Next consider a constant portfolio in the interval \([0, T]\) which is invested in \(N\) standard European derivatives with pay-off functions \(g_1, \ldots, g_N\) and time of maturity \(T\). Let the price of these derivatives be given by the Black-Scholes formula. Show that the portfolio satisfies the dominance principle (max. 2 points).

**Solution:** See Borell’s notes, Theorems 5.2.1 and 5.1.2

3. Consider two stocks with prices

\[ S_1(t) = S_1(0)e^{\left(\mu_1 - \frac{\sigma_1^2}{2}\right)t + \sigma_1 W_1(t)} \quad S_2(t) = S_2(0)e^{\left(\mu_2 - \frac{\sigma_2^2}{2}\right)t + \sigma_2 W_2(t)} \]

where \(\sigma_1, \sigma_2 > 0\), \(\mu_1, \mu_2 \in \mathbb{R}\) and \(W_1(t), W_2(t)\) are two Brownian motions. Let \(T > 0\) and assume that \(W_1(T)\) and \(W_2(T)\) are independent random variables. Compute the Markowitz portfolio of an investor with initial capital \(K > 0\) and risk aversion \(\theta\) who wants to invest in the stocks and in a money market with interest \(r > 0\) during the interval of time \([0, T]\) (max. 5 points)

**Solution:** Let \(a_1\) be the number of shares of the first stock and \(a_2\) the number of shares of the second stock in the Markowitz portfolio and let

\[ \pi_1 = \frac{a_1 S_1(0)}{K}, \quad \pi_2 = \frac{a_2 S_2(0)}{K}. \]

Then we have the formula

\[ \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix} = \frac{1}{2\theta} C^{-1} \begin{pmatrix} \mathbb{E}[R_1] - \rho \\ \mathbb{E}[R_2] - \rho \end{pmatrix} \]  \hspace{1cm} (2)

where \(R_1, R_2\) are the returns of the two stocks, \(\rho = e^{rT} - 1\) and \(C^{-1}\) is the inverse of the covariant matrix \(C_{ij} = C(R_i, R_j)\). The random variables \(R_1, R_2\), are given by

\[ R_1 = \frac{S_1(T) - S_1(0)}{S_1(0)}, \quad R_2 = \frac{S_2(T) - S_2(0)}{S_2(0)}. \]

It follows that

\[ \mathbb{E}[R_1] = \frac{1}{S_1(0)} \mathbb{E}[S_1(T)] - 1, \quad \text{Var}[R_1] = \frac{1}{S_1(0)^2} \text{Var}[S_1(T)] \]

and similarly for the second stock. Since the expectation and the variance of the geometric Brownian motion \(S_1(t)\) are given by

\[ \mathbb{E}[S_1(t)] = S_1(0)e^{\mu_1 t}, \quad \text{Var}[S_1(t)] = S_1(0)^2 e^{2\mu_1 t} (e^{\sigma_1^2 t} - 1) \]

we obtain

\[ \mathbb{E}[R_1] = e^{\mu_1 T} - 1, \quad \text{Var}[R_1] = e^{2\mu_1 T} (e^{\sigma_1^2 T} - 1). \]

and similarly for the second stock

\[ \mathbb{E}[R_2] = e^{\mu_2 T} - 1, \quad \text{Var}[R_2] = e^{2\mu_2 T} (e^{\sigma_2^2 T} - 1). \]
Since $W_1(T)$ and $W_2(T)$ are independent, then $R_1$ and $R_2$ are also independent. Therefore $\text{Cov}(R_1, R_2) = C_{12} = C_{21} = 0$ and so the matrix of covariances of the random variables $R_1, R_2$ is given by

$$C = \begin{pmatrix} \text{Var}[R_1] & 0 \\ 0 & \text{Var}[R_2] \end{pmatrix} \Rightarrow C^{-1} = \begin{pmatrix} 1/\text{Var}[R_1] & 0 \\ 0 & 1/\text{Var}[R_2] \end{pmatrix}$$

Replacing in (2) we obtain

$$\pi_1 = \frac{1}{2\theta} \frac{\text{E}[R_1] - \rho}{\text{Var}[R_1]} = \frac{1}{2\theta} \frac{e^{\mu_1 T} - e^{rT}}{e^{2\mu_1 T} (e^{\sigma_1^2 T} - 1)}, \quad \pi_2 = \frac{1}{2\theta} \frac{\text{E}[R_2] - \rho}{\text{Var}[R_2]} = \frac{1}{2\theta} \frac{e^{\mu_2 T} - e^{rT}}{e^{2\mu_2 T} (e^{\sigma_2^2 T} - 1)}$$

This completes the exercise (5 points).