1. Consider an American derivative with intrinsic value

\[ Y(t) = \min(S(t), (24 - S(t))_+) \]

and expiring at time \( T = 3 \). The initial price of the underlying stock is \( S(0) = 27 \), while at future times it follows the binomial model

\[
S(t + 1) = \begin{cases} 
\frac{4}{3}S(t) & \text{with probability } 1/2 \\
\frac{2}{3}S(t) & \text{with probability } 1/2
\end{cases}
\]

for \( t = 0, 1, 2 \). Assume also that the interest rate of the bond is zero. Compute the possible paths of the fair value of the derivative (max. 2 points). In which case it is optimal for the buyer to exercise the derivative prior to expiration? (max. 1 point). Compute the expected return of a constant portfolio which consists of a long position in one share of the stock and a short position in one share of the derivative (max. 2 points).

Solution: With the given values of the parameters \( u, d, r \), we have

\[
q_u = \frac{e^r - e^d}{e^u - e^d} = \frac{1 - \frac{2}{3}}{\frac{4}{3} - \frac{2}{3}} = \frac{1}{2} = q_d.
\]

The fair price \( \hat{\Pi}_Y(t) \) of the American derivative satisfies

\[
\hat{\Pi}_Y(t) = \max(Y(t), e^{-r}(q_u\hat{\Pi}_Y^u(t + 1) + q_d\hat{\Pi}_Y^d(t + 1)))
\]

\[
= \max(Y(t), \frac{1}{2}(\hat{\Pi}_Y^u(t + 1) + \hat{\Pi}_Y^d(t + 1))),
\]

where \( \hat{\Pi}_Y^u(t) \) (resp. \( \hat{\Pi}_Y^d(t) \)) is the price of the derivative at time \( t \) assuming that the stock price goes up (resp. down) at time \( t \). The diagram of the stock price is
to which there corresponds the following diagram for the intrinsic value:

\[
\begin{align*}
Y(3) &= 0 = \hat{\Pi}_Y(3) \\
Y(2) &= 0 \\
Y(1) &= 0 \\
Y(0) &= 0 \\
Y(1) &= 6 \\
Y(2) &= 12 \\
Y(3) &= 8 = \hat{\Pi}_Y(3)
\end{align*}
\]

Therefore

\[
S(2) = 48 \Rightarrow \hat{\Pi}_Y(2) = 0,
S(2) = 24 \Rightarrow \hat{\Pi}_Y(2) = 4,
S(2) = 12 \Rightarrow \hat{\Pi}_Y(2) = 12
\]
$S(1) = 36 \Rightarrow \hat{\Pi}_Y(1) = 2, \quad S(1) = 18 \Rightarrow \hat{\Pi}_Y(1) = 8,$

and $\hat{\Pi}_Y(0) = 5$. We thereby obtained the following diagram for the price of the derivative:

This completes the first part of the exercise (2 points). The only case in which the price of the derivative equals its intrinsic value prior to expiration is at time $t = 2$ when the price of the stock is $S(2) = 12$ (i.e., the stock price goes down in the first two steps). This is indicated in the previous diagram by putting the price of the derivative in a box. In this case, and only in this case, it is optimal to exercise the derivative prior to expiration. This completes the second part of the exercise (1 point). Let $V(t)$ be the value of a portfolio with 1 share of the stock and -1 share of the derivative. The return of the portfolio is the random variable

$$R = \frac{V(3) - V(0)}{V(0)} = \frac{V(3)}{V(0)} - 1.$$  

The expected return is

$$\mathbb{E}[R] = \frac{1}{V(0)} \mathbb{E}[V(3)] - 1.$$  

We have

$$V(0) = S(0) - \hat{\Pi}_Y(0) = 22$$
and

\[ V(3) = \begin{cases} 
64 \text{ with probability } \frac{1}{8} \\
32 \text{ with probability } \frac{3}{8} \\
8 \text{ with probability } \frac{3}{8} \\
0 \text{ with probability } \frac{1}{8} 
\end{cases} \]

Hence \( \mathbb{E}[V(3)] = 64 \cdot \frac{1}{8} + 32 \cdot \frac{3}{8} + 8 \cdot \frac{3}{8} + 0 \cdot \frac{1}{8} = 23 \). Therefore

\[ \mathbb{E}[R] = \frac{23}{22} - 1 = \frac{1}{22} \approx 4.54\% , \]

which completes the third part of the exercise (2 points).

2. Consider a European derivative with pay-off \( Y = S(T)(S(T) - K) \) and time of maturity \( T \), where \( K > 0 \) is a constant. It is assumed that the price \( S(t) \) of the underlying stock follows a geometric Brownian motion for \( t \in [0, T] \) and that the interest rate of the bond is a constant \( r > 0 \). Compute the Black-Scholes price \( \Pi_Y(t) \) of this derivative (max. 2 points) and the hedging portfolio \( h(t) = (h_S(t), h_B(t)) \) (max. 1 point). Finally, assume \( S(0) = K \) and compute the expected return of a constant portfolio with 1 share of this derivative (max. 2 points)

**Solution:** The pay-off function is \( g(z) = z(z - K) \); the Black-Scholes price is given by \( \Pi_Y(t) = v(t, S(t)) \), where

\[ 
v(t, s) = e^{-rt} \int_{-\infty}^{\infty} g(se^{(r-\frac{\sigma^2}{2})\tau - \sigma \sqrt{\tau} x}) e^{-\frac{x^2}{2\tau}} \frac{dx}{\sqrt{2\pi}} \\
= e^{-rt} \int_{-\infty}^{\infty} \left[ se^{(r-\frac{\sigma^2}{2})\tau - \sigma \sqrt{\tau} x} - K \right] e^{-\frac{x^2}{2\tau}} \frac{dx}{\sqrt{2\pi}} \\
= s \left[ se^{(r+\sigma^2)\tau} - K \right]. \]

This solves the first part of the exercise (2 points). The number of shares of the underlying stock in the hedging portfolio is given by \( h_S(t) = \Delta(t, S(t)) \), where

\[ \Delta(t, s) = \frac{\partial v}{\partial s} = 2se^{(r+\sigma^2)\tau} - K. \]

The number of shares of the bond is obtained using

\[ \Pi_Y(t) = h_S(t)S(t) + h_B(t)B(t) \Rightarrow h_B(t) = \frac{1}{B(t)}(\Pi_Y(t) - h_S(t)S(t)) = -\frac{e^{-rt}}{B(0)}e^{(r+\sigma^2)\tau}S(t)^2. \]

This completes the second part of the exercise (1 point). The return of the portfolio is

\[ R = \frac{\Pi_Y(T)}{\Pi_Y(0)} - 1. \]
Using \( Y(T) = S(T)(S(T) - K) \) and \( Y(0) = K^2(e^{(r+\sigma^2)T} - 1) \), we obtain

\[
\mathbb{E}[R] = \frac{\mathbb{E}[S(T)Y(T)]}{K^2(e^{(r+\sigma^2)T} - 1)} - 1.
\]

Writing the geometric Brownian motion at time \( T \) as

\[
S(T) = S(0)e^{(\mu - \frac{\sigma^2}{2})T + \sigma W(T)} = Ke^{(\mu - \frac{\sigma^2}{2})T + \sigma \sqrt{T}G},
\]

where \( G = W(T)/\sqrt{T} \in N(0,1) \), we get

\[
\mathbb{E}[S(T)^2] = K^2e^{(2\mu - \sigma^2)T} \int e^{2\sigma \sqrt{T}x - \frac{1}{2}x^2} \frac{dx}{\sqrt{2\pi}} = K^2e^{(2\mu + \sigma^2)T},
\]

\[
\mathbb{E}[S(T)] = Ke^{(\mu - \frac{\sigma^2}{2})T} \int e^{\sigma \sqrt{T}x - \frac{1}{2}x^2} \frac{dx}{\sqrt{2\pi}} = Ke^{\mu T}.
\]

Therefore

\[
\mathbb{E}[S(T)(S(T) - K)] = \mathbb{E}[S(T)^2] - K\mathbb{E}[S(T)] = K^2e^{\mu T}(e^{(\mu + \sigma^2)T} - 1).
\]

We conclude that the expected return is given by

\[
\mathbb{E}[R] = \frac{e^{\mu T}(e^{(\mu + \sigma^2)T} - 1)}{e^{(r+\sigma^2)T} - 1} - 1.
\]

This completes the third part of the exercise (2 points).

3. Assume that the dominance principle holds. Let \( p(t, S(t), K, T) \) be the value at time \( t \) of a European put option with strike price \( K \) and maturity time \( T \) on a stock with price \( S(t) \). Let \( c(t, S(t), K, T) \) be the price of the corresponding European call option, while \( C(t, S(t), K, T) \) and \( P(t, S(t), K, T) \) denote the price of the American call and put with the same parameters. Give a complete proof of the following facts: (i) The put-call parity (max. 1 point); (ii) The price of the European call is a non-decreasing function of the time of maturity (max. 2 points); (iii) \( C(t, S(t), K, T) > S(t) - K \), for \( t \in [0, T] \) (max. 1 point). Explain why (iii) implies that the American and the European call have the same value (max. 1 point).

**Solution:** See Borell’s book *Introduction to the Black-Scholes theory*, Theorems 1.1.2, 1.1.3.