SOLUTIONS
OPTIONS AND MATHEMATICS
(CTH[$mve095$], GU[$MMA700$]
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No aids.
Questions on the exam: Peter Helgesson, tel 070-65 77 412
Each problem is worth 3 points.

1. (Black-Scholes Model) A simple European-style derivative pays the amount

$$Y = (S(T) - S(0))^+$$

at time of maturity T. Find the time zero price $\Pi_Y(0)$ of the derivative if the stock pays the dividend $(1-e^{-rT})S(\frac{T}{2}-)$ at time $\frac{T}{2}$.

Solution. Set $s = S(0)$, $g(x) = (x - S(0))^+$ if $x > 0$, and $\delta = 1 - e^{-rT}$. It is kown that

$$\Pi_Y(0) = e^{-rT} E\left[g((1-\delta)se^{(r-\frac{\sigma^2}{2})T+\sigma W(T)}) \right]$$

$$= c(0,(1-\delta)s,s,T) = (1-\delta)s\Phi(\frac{\ln\frac{(1-\delta)s}{s} + (r+\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}) - se^{-rT}\Phi(\frac{\ln\frac{(1-\delta)s}{s} + (r-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}})$$

$$= se^{-rT}\left(\Phi(\frac{\sigma\sqrt{T}}{2}) - \Phi(-\frac{\sigma\sqrt{T}}{2})\right)$$

$$= S(0)e^{-rT}\left(2\Phi(\frac{\sigma\sqrt{T}}{2}) - 1\right).$$

2. Let $(X_k)_{k=1}^n$ be an i.i.d., where X_1 possesses the probability density

$$\frac{1}{2\sqrt{2\pi}}(1+x+x^2)e^{-\frac{x^2}{2}}, -\infty < x < \infty.$$

Find the characteristic function of $S_n = X_1 + ... + X_n$.

Solution. For each real ξ,

$$c_{S_n}(\xi) = E\left[e^{i\xi S_n}\right] = E\left[\prod_{k=1}^{n} e^{i\xi X_k}\right]$$

$$= \prod_{k=1}^{n} E\left[e^{i\xi X_k}\right] = \left(E\left[e^{i\xi X_1}\right]\right)^n.$$

Moreover,

$$E\left[e^{i\xi X_1}\right] = \int_{-\infty}^{\infty} e^{i\xi x}(1+x+x^2)e^{-\frac{x^2}{2}} \frac{dx}{2\sqrt{2\pi}}$$

$$= \frac{1}{2}\left(\int_{-\infty}^{\infty} e^{i\xi x}e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} + \int_{-\infty}^{\infty} e^{i\xi x}xe^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} + \int_{-\infty}^{\infty} e^{i\xi x}x^2e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}}\right).$$

Here

$$\int_{-\infty}^{\infty} e^{i\xi x}e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} = e^{-\frac{\xi^2}{2}},$$

$$\int_{-\infty}^{\infty} e^{i\xi x}xe^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} = -i\frac{d}{d\xi}\int_{-\infty}^{\infty} e^{i\xi x}e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} = i\xi e^{-\frac{\xi^2}{2}},$$

and

$$\int_{-\infty}^{\infty} e^{i\xi x}x^2e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} = -i\frac{d}{d\xi}\int_{-\infty}^{\infty} e^{i\xi x}xe^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} = (1-\xi^2)e^{-\frac{\xi^2}{2}}.$$

Hence

$$c_{S_n}\xi) = (1 + \frac{i}{2}\xi - \frac{1}{2}\xi^2)^n e^{-\frac{n\xi^2}{2}}.$$

3. (Black-Scholes Model) The joint stock price process $S = (S_1(t), S_2(t))_{t\geq 0}$ is governed by a bivariate geometric Brownian motion with volatility (σ_1, σ_2) and correlation ρ.

A European-style derivative pays the amount

$$Y = \frac{(S_2(T) - S_1(T))^2}{\sqrt{S_1(T)S_2(T)}}$$

at time of maturity T. Find the time zero price $\Pi_Y(0)$ of the derivative.

Solution. Note that $Y = g(S_1(T), S_2(T))$, where the function

$$g(x_1, x_2) = \frac{(x_2 - x_1)^2}{\sqrt{x_1 x_2}}$$

is positively homogenous of degree one. Therefore, let S_2 be a numéraire and put

$$S = \frac{S_1}{S_2}$$

where S is a geometric Brownan motin with volatility

$$\sigma_- = \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}.$$

Moreover, let

$$s = \frac{S_1(0)}{S_2(0)}$$

and recall that

$$\frac{Y}{S_2(T)} = \frac{(1 - S(T))^2}{\sqrt{S(T)}} = g(S(T), 1).$$

Now with S_2 as a numéraire, by applying the Black-Scholes theory with $r = 0$, we concude that the derivate has the time zero price

$$E\left[g(se^{-\frac{\sigma_-^2}{2}T + \sigma_- W(T)}, 1)\right]$$

$$= E\left[s^{-\frac{1}{2}} e^{\frac{\sigma_-^2}{4}T - \frac{\sigma_-}{2}W(T)} - 2s^{\frac{1}{2}} e^{-\frac{\sigma_-^2}{4}T + \frac{\sigma_-}{2}W(T)} + s^{\frac{3}{2}} e^{-\frac{3\sigma_-^2}{4}T + \frac{3\sigma_-}{2}W(T)}\right]$$

$$= s^{-\frac{1}{2}} e^{\frac{\sigma_-^2}{2}T} - 2s^{\frac{1}{2}} + s^{\frac{3}{2}} e^{\frac{3\sigma_-^2}{2}T}.$$

In the original price unit we get the time-zero price

$$S_1^{-1/2}(0) S_2^{3/2}(0) e^{\frac{\sigma_-^2}{2}T} - 2S_1^{1/2}(0) S_2^{1/2}(0) + S_1^{3/2}(0) S_2^{-1/2}(0) e^{\frac{3\sigma_-^2}{2}T}.$$

4. Show that there exists an arbitrage portfolio in the single-period binomial model if and only if

$$r \notin \left]d, u\right[.$$

5. (Black-Scholes Model) Assume $t, T \in \mathbf{R}$, $\tau = T - t > 0$, and $g \in \mathcal{P}$.

(a) Define the price $\Pi_Y(t)$ at time t of a European derivative with payoff $g(S(T))$ at time of maturity T.

(b) Let

$$d_1 = \frac{1}{\sigma\sqrt{\tau}}(\ln\frac{s}{K} + (r + \frac{\sigma^2}{2})\tau),$$

and $d_2 = d_1 - \sigma\sqrt{\tau}$. Show that

$$c(t, s, K, T) = s\Phi(d_1) - Ke^{-r\tau}\Phi(d_2).$$