SOLUTIONS
OPTIONS AND MATHEMATICS
(CTH[mve095], GU[MMA700])
May 27, 2013, morning, v
No aids.
Questions on the exam: Christer Borell, telephone number 0705 292322
Each problem is worth 3 points.

1. (Binomial Model: \(S(0) = B(0) = 1\), \(T = 2\), \(u = -d = \ln 2\), and \(r = 0\)). A European-style financial derivative pays the amount \(Y\) at time of maturity \(T = 2\), where

\[
Y = \begin{cases} 
0 & \text{if } S(0) = S(2), \\
S(1) & \text{if } S(0) \neq S(2).
\end{cases}
\]

(a) State the time zero price \(\Pi_Y(0)\) of the derivative.
(b) The portfolio strategy \(h\) replicates \(Y\). State \(h(0) = (h_S(0), h_B(0))\).

(Please, do not hand in any solutions, just answers!)

Solution. (a) We have

\[
q_u = \frac{e^r - e^d}{e^u - e^d} = \frac{1}{3}
\]

and

\[
q_d = \frac{e^u - e^r}{e^u - e^d} = \frac{2}{3}.
\]

Moreover, if \(v(t) = \Pi_Y(t)\) and \(s = S(0)\),

\[
\begin{align*}
  v(2)|_{X_1=u,X_2=u} &= se^u \\
  v(2)|_{X_1=u,X_2=d} &= 0 \\
  v(2)|_{X_1=d,X_2=u} &= 0 \\
  v(2)|_{X_1=d,X_2=d} &= se^d
\end{align*}
\]

and, hence,

\[
\begin{align*}
  v(1)|_{X_1=u} &= e^{-r}q_u se^u = q_u se^{u-r} \\
  v(1)|_{X_1=d} &= e^{-r}q_d se^d = q_d se^{d-r}.
\end{align*}
\]

Now

\[
\Pi_Y(0) = e^{-r}(q_u^2 se^{u-r} + q_d^2 se^{d-r}) = se^{-2r}(q_u^2 e^u + q_d^2 e^d) = \frac{4}{9}s = \frac{4}{9}.
\]
(c) We have
\[
\begin{align*}
& h_S(0)se^u + h_B(0)B(0)e^r = q_u se^{u-r} \\
& h_S(0)se^d + h_B(0)B(0)e^r = q_d se^{d-r}.
\end{align*}
\]
Thus
\[
h_S(0) = e^{-r} \frac{q_u e^u - q_d e^d}{e^u - e^d} = \frac{2}{9}
\]
and
\[
h_B(0) = s \frac{e^{u+d-2r}}{B(0)} \frac{q_d - q_u}{e^u - e^d} = \frac{2}{9} s = \frac{2}{9}.
\]

2. (Black-Scholes Model) A European-style financial derivative has at time zero the price $a$ and pays the amount
\[
Y = \begin{cases} 
a + \xi & \text{if } S(T) \geq S(0) \\
a & \text{if } S(T) < S(0).\end{cases}
\]
at time of maturity $T$, where $a$ and $T$ are given positive numbers and $\xi$ is an unknown real number. Find $\xi$.

Solution. Put $Z = H(S(T) - S(0))$, where $H$ is the Heaviside function. Now $Y = a + \xi Z$ and
\[
a = ae^{-rT} + \xi \Pi_Z(0).
\]
Thus
\[
\xi = \frac{a(1 - e^{-rT})}{\Pi_Z(0)}.
\]
Moreover, if $s = S(0)$,
\[
\Pi_Z(0) = e^{-rT} E \left[ H(s(e^{(r-\frac{\sigma^2}{2})T-\sigma\sqrt{T}G}) - 1) \right],
\]
where $G \in N(0, 1)$ and, hence,
\[
\Pi_Z(0) = e^{-rT} P \left[ G \leq \left( \frac{r}{\sigma} - \frac{\sigma}{2} \right) \sqrt{T} \right] = e^{-rT} \Phi \left( \frac{r}{\sigma} - \frac{\sigma}{2} \sqrt{T} \right).
\]
Summing up, we have
\[
\xi = \frac{a(e^{-rT} - 1)}{\Phi \left( \frac{r}{\sigma} - \frac{\sigma}{2} \sqrt{T} \right)}.
\]
3. (Black-Scholes Model) Suppose $K, T > 0$ and $N \in \mathbb{N}_+$ are given and consider a European-style derivative which pays the amount

$$Y = \left( \prod_{j=1}^{N} S\left(\frac{jT}{N}\right)^{\frac{1}{N}} - K \right)^+$$

at time of maturity $T$. Find the time zero price $\Pi_Y(0)$ of the derivative. (Hint: $1^2 + 2^2 + ... + N^2 = \frac{1}{6}N(N+1)(2N+1)$)

Solution. Set $S(0) = s$ to get

$$\Pi_Y(0) = e^{-rT} E \left[ \left( s \left( \prod_{j=1}^{N} e^{(r-\frac{\sigma^2}{2})\frac{jT}{N} + \sigma W\left(\frac{jT}{N}\right)} \right)^{\frac{1}{N}} - K \right)^+ \right]$$

$$= e^{-rT} E \left[ \left( s e^{(r-\frac{\sigma^2}{2})\frac{1}{2N} + \frac{\sigma^2}{N} \sum_{j=1}^{N} W\left(\frac{jT}{N}\right)} - K \right)^+ \right].$$

Set $X = \sum_{k=1}^{N} W\left(\frac{kT}{N}\right)$. Clearly, $X$ is a centred Gaussian random variable and to find its variance put

$$Z_j = W\left(\frac{jT}{N}\right) - W\left(\frac{(j-1)T}{N}\right), \quad j = 1, ..., N.$$

Then

$$X = \sum_{j=1}^{N} \left( \sum_{i=1}^{k} Z_i \right) = \sum_{1 \leq i \leq j \leq N} Z_i = \sum_{i=1}^{N} (N - i + 1)Z_i$$

and

$$\text{Var}(X) = \sum_{i=1}^{N} (N - i + 1)^2 \text{Var}(Z_i)$$

$$= \frac{T}{N} \sum_{i=1}^{N} (N - i + 1)^2 = \frac{T}{6}N(N+1)(2N+1).$$
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Thus
\[ \Pi_Y(0) = e^{-rT} E \left[ \left( se^{(r-\frac{\sigma^2}{2})(N+1)T} + \frac{\sigma}{\pi} \sqrt{\frac{(N+1)(2N+1)}{6}} G - K \right)^+ \right], \]

where \( G \in N(0,1) \). Now put
\[
\begin{align*}
    a &= (r - \frac{\sigma^2}{2}) \frac{(N+1)T}{2N} \\
    b &= \frac{\sigma}{N} \sqrt{\frac{(N+1)(2N+1)T}{6}}
\end{align*}
\]

so that
\[
\Pi_Y(0) = e^{-rT} E \left[ \left( se^{a-bG} - K \right)^+ \right]
\]

\[ = e^{-rT} \left( se^a \int_{-\infty}^c e^{-bx - \frac{1}{2}x^2} \frac{dx}{\sqrt{2\pi}} - K \int_{-\infty}^c e^{-bx - \frac{1}{2}x^2} \frac{dx}{\sqrt{2\pi}} \right) \]

where
\[ c = \ln \frac{s}{K} + a \]

Summing up, we get
\[ \Pi_Y(0) = e^{-rT} \left( se^{a + \frac{b^2}{2}} \Phi(c + b) - K \Phi(c) \right) \]

with \( a, b, \) and \( c \) defined as above.

4. Let \((X_n)_{n=1}^{\infty}\) be an i.i.d. such that \( P[X_1 = 1] = P[X_1 = -1] = \frac{1}{2} \) and set
\[ Y_n = \frac{1}{\sqrt{n}} (X_1 + \ldots + X_n), \ n \in \mathbb{N}_+. \]

Prove that \( Y_n \to G \), where \( G \in N(0,1) \).

5. (Dominance Principle) Show that the map
\[ K \to c(t, S(t), K, T), \ K > 0 \]

is convex.