1. (Dominance Principle) Consider a forward contract on $S$ with delivery date $T$ and denote by $K$ the forward price $S^T_{for}(0)$ at time zero. A European-style call on $S$ with strike price $K$ and time of maturity $T$ has the time zero price $a$ crowns. Find the time zero price of a European-style put with strike price $K$ and time of maturity $T$.

Solution. By the Put-Call Theorem

$$S(0) - c(0, S(0), K, T) = Ke^{-rT} - p(0, S(0), K, T)$$

and

$$K = S^T_{for}(0) = S(0)e^{rT}.$$ 

Now as

$$c(0, S(0), K, T) = a$$

we have

$$p(0, S(0), K, T) = a.$$ 

2. (Black-Scholes model) Consider a European-style derivative paying the amount

$$Y = \sqrt{S(T/2)S(T)}$$

at time of maturity $T$. Find $\Pi_Y(t)$ if $0 \leq t < T$.

Solution.
Case $T/2 < t < T$. Let $\tau = T - t$, $s = S(t)$, and $G \in N(0, 1)$. Since $a = S(T/2)$ is known,

$$
\Pi_Y(t) = e^{-\tau r} E \left[ \sqrt{a se^{(r - \frac{a^2}{2})\tau + \sigma \sqrt{\tau} G}} \right]
$$

$$
= \sqrt{a se^{(r - \frac{a^2}{2})\tau}} e^{-\tau r} E \left[ e^{\frac{1}{2} \sqrt{\tau} G} \right]
$$

$$
= \sqrt{a se^{\frac{1}{2}(r - \frac{a^2}{2})\tau - \frac{a^2}{2} r}} = \sqrt{a se^{-\frac{1}{2} r + \frac{a^2}{2} r}}
$$

$$
= b(t) \sqrt{S(t)}
$$

where

$$
b(t) = \sqrt{S(T/2)} e^{-(\frac{1}{2} r + \frac{a^2}{2}) (T - t)}.
$$

Case $0 \leq t < T/2$. We have

$$
\Pi_Y(T/2) = S(T/2) e^{-(\frac{1}{2} r + \frac{a^2}{16}) T}
$$

and, hence,

$$
\Pi_Y(t) = S(t) e^{-(\frac{1}{2} r + \frac{a^2}{16}) T}.
$$

3. Let $Z(t) = (Z_1(t), Z_2(t))$, $t \geq 0$, be a standard Brownian motion in the plane. Find

$$
\text{Var}(\max(Z_1(t), Z_2(t))).
$$

Solution. Set $X = \max(Z_1(t), Z_2(t))$. Since $Z_1(0) = Z_2(0) = 0$, $\text{Var}(X) = 0$ if $t = 0$. Next assume $t > 0$. The random variables $Z_1(t), Z_2(t) \in N(0, t)$ are independent and

$$
P[X \leq x] = P[Z_1(t) \leq x] P[Z_2(t) \leq x] = \Phi^2(x / \sqrt{t}).
$$

Hence

$$
f_X(x) = 2\varphi(x / \sqrt{t}) \Phi(x / \sqrt{t}) / \sqrt{t}
$$

and

$$
E[X] = \int_{-\infty}^{\infty} x f_X(x) dx = 2\sqrt{t} \int_{-\infty}^{\infty} x \varphi(x) \Phi(x) dx
$$
\[= 2\sqrt{t} \left\{ [-\varphi(x)\Phi(x)]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \varphi^2(x)dx \right\}\]
\[= 2\sqrt{t} \int_{-\infty}^{\infty} \varphi^2(x)dx\]
\[= 2\sqrt{t} \int_{-\infty}^{\infty} \varphi(\sqrt{2x})dx/\sqrt{2\pi} = \sqrt{t/\pi}.\]

Moreover,
\[E\left[X^2\right] = \int_{-\infty}^{\infty} x^2 f_X(x)dx = 2t \int_{-\infty}^{\infty} x^2 \varphi(x)\Phi(x)dx\]
\[= 2t \left\{ [-\varphi(x)x\Phi(x)]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \varphi(x)(\Phi(x) + x\varphi(x))dx \right\}\]
\[= 2t \int_{-\infty}^{\infty} \varphi(x)(\Phi(x) + x\varphi(x))dx\]
\[= 2t \int_{-\infty}^{\infty} \varphi(x)\Phi(x)dx = t.\]

Thus
\[\text{Var}(X) = t(1 - \frac{1}{\pi}),\]
a formula which also holds if \(t = 0\).

4. Consider a single-period binomial model and assume \(d < r < u\). A derivative pays the amount \(Y = f(X)\) at time 1, where \(X = \ln(S(1)/S(0))\). Find a portfolio \(h = (h_S, h_B)\) which replicates \(Y\).

5. (Black-Scholes model) Suppose \(0 \leq t < T < \infty\). A simple European-style derivative paying the amount \(Y = g(S(T))\) at time of maturity \(T\) has the price \(v(t, S(t))\) at time \(t\), where
\[v(t, s) = e^{-r(T-t)}E\left[g(se^{r\frac{\sigma^2}{2}}(T-t)+\sigma\sqrt{T-t}G)\right]\]
and \(G \sim N(0,1)\). Find the time-\(t\) price of a European-style call with strike price \(K\) and time of maturity \(T\).