1. (Binomial model with $T = 2$, $u = -d > 0$, and $e^r = \frac{1}{2}(e^u + e^d)$) A derivative of European type pays the amount

$$Y = \left| \frac{S(T)}{S(0)} - 1 \right|$$

at time of maturity $T$. Find $\Pi_Y(0)$.

Solution. We have that

$$q_u = \frac{e^r - e^d}{e^u - e^d} = \frac{1}{2}(e^u + e^d) - e^d = \frac{1}{2}$$

and if $v(t) = \Pi_Y(t)$,

$$\begin{cases} 
  v(2) | X_1 = u, X_2 = u = e^{2u} - 1 \\
  v(2) | X_1 = u, X_2 = d = 0 \\
  v(2) | X_1 = d, X_2 = u = 0 \\
  v(2) | X_1 = d, X_2 = d = 1 - e^{-2u}.
\end{cases}$$

Now

$$\begin{cases} 
  v(1) | X_1 = u = \frac{e^{-r}}{2}(e^{2u} - 1) \\
  v(1) | X_1 = d = \frac{e^{-r}}{2}(1 - e^{-2u})
\end{cases}$$

and

$$\Pi_Y(0) = e^{-r} \left( \frac{e^{-r}}{4}(e^{2u} - 1) + \frac{e^{-r}}{4}(1 - e^{-2u}) \right)$$

$$= e^{-2r} \left( \frac{e^{2u} - e^{-2u}}{4} \right) = \frac{e^{2u} - e^{-2u}}{(e^u + e^{-u})^2} = \frac{e^u - e^{-u}}{e^u + e^{-u}}.$$
2. Let $Z(t) = (Z_1(t), Z_2(t)), t \geq 0$, be a standard Brownian motion in the plane. Find

$$\text{Var}(e^{Z_1(t)} - e^{Z_2(t)}).$$

Solution. The random variables $Z_1(t), Z_2(t) \in N(0,t)$ are independent and

$$E \left[ e^{aG} \right] = e^{a^2}$$
if $G \in N(0,1)$ and $a \in \mathbb{R}$. Accordingly from these properties,

$$E \left[ e^{Z_1(t)} - e^{Z_2(t)} \right] = e^t - e^t = 0$$

and

$$E \left[ (e^{Z_1(t)} - e^{Z_2(t)})^2 \right] = E \left[ e^{2Z_1(t)} \right] - 2E \left[ e^{Z_1(t)} e^{Z_2(t)} \right] + E \left[ e^{2Z_2(t)} \right]$$

$$= 2e^{2t} - 2E \left[ e^{Z_1(t)} \right] E \left[ e^{Z_2(t)} \right] = 2e^{2t} - 2e^t.$$

The above formulas give

$$\text{Var}(e^{Z_1(t)} - e^{Z_2(t)}) = 2e^t(e^t - 1).$$

Alternative solution. Since $e^{Z_1(t)}$ and $-e^{Z_2(t)}$ are independent

$$\text{Var}(e^{Z_1(t)} - e^{Z_2(t)}) = \text{Var}(e^{Z_1(t)}) + \text{Var}(-e^{Z_2(t)})$$

$$= 2\text{Var}(e^{Z_1(t)}) = 2(E \left[ e^{2Z_1(t)} \right] - (E \left[ e^{Z_1(t)} \right])^2)$$

$$= 2e^{2t} - 2e^t = 2e^t(e^t - 1).$$

3. (Black-Scholes model) A stock price process $(S(t))_{t \geq 0}$ is governed by the equation

$$S(t) = S(0)e^{(\mu - \frac{\sigma^2}{2})t + \sigma W(t)}, \; t \geq 0,$$

where $\mu > r$. If $T$ and $K$ denote strictly positive real numbers, show that

$$E \left[ (S(T) - K)^+ \right] > e^{rT} c(0, S(0), K, T).$$
Solution. If $a > 0$ and

\[ f(x, a) = (S(0)e^{(a-\frac{\sigma^2}{2})T-\sigma\sqrt{T}x} - K)^+, \quad x \in \mathbb{R}, \]

then

\[ E \left[ (S(T) - K)^+ \right] = \int_{-\infty}^{\infty} f(x, \mu)\varphi(x)dx \]

and

\[ e^{\mu T}c(0, S(0), K, T) = \int_{-\infty}^{\infty} f(x, r)\varphi(x)dx. \]

Since $\mu > r$ we have $f(x, \mu) \geq f(x, r)$ with strict inequality if

\[ x < \frac{1}{\sigma\sqrt{T}} \left( \ln \frac{S(0)}{K} + (\mu - \frac{\sigma^2}{2})T \right). \]

Hence

\[ \int_{-\infty}^{\infty} f(x, \mu)\varphi(x)dx > \int_{-\infty}^{\infty} f(x, r)\varphi(x)dx \]

which proves that

\[ E \left[ (S(T) - K)^+ \right] > e^{\mu T}c(0, S(0), K, T). \]

Alternative solution. For any strictly positive real number $a$ the Black-Scholes theory yields

\[ f(a) = \text{def} \quad e^{-aT}\int_{-\infty}^{\infty} (S(0)e^{(a-\frac{\sigma^2}{2})T-\sigma\sqrt{T}x} - K)^+dx \]

\[ = S(0)\Phi(d_1(a)) - Ke^{-aT}\Phi(d_2(a)) \]

where

\[ d_1(a) = \frac{\ln \frac{S(0)}{K} + (a + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \]

and

\[ d_2(a) = \frac{\ln \frac{S(0)}{K} + (a - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} = d_1(a) - \sigma\sqrt{T}. \]
Hence
\[ f'(a) = S(0)\varphi(d_1(a)) \frac{1}{\sigma \sqrt{T}} - Ke^{-aT}\varphi(d_2(a)) \frac{1}{\sigma \sqrt{T}} + KTe^{-aT}\Phi(d_2(a)). \]

But
\[ S(0)\varphi(d_1(a)) - Ke^{-aT}\varphi(d_2(a)) = 0 \]
(cf the proof of Theorem 5.3.1 in the textbook) and we get
\[ f'(a) = KTe^{-aT}\Phi(d_2(a)). \]

Hence
\[ \frac{d}{da} (e^{aT} f(a)) = Te^{aT} f(a) + KT\Phi(d_2(a)) > 0 \]
and if \( \mu > r \), we get
\[ E \left[ (S(T) - K)^+ \right] = e^{\mu T} f(\mu) > e^{r T} f(r) = e^{r T} c(0, S(0), K, T). \]

4. (Dominance principle) Show that the map
\[ K \rightarrow c(t, S(t), K, T), \ K > 0 \]
is convex.

5. (Black-Scholes model) Suppose \( K, T, \) and \( \sigma \) are strictly positive real numbers.

(a) Let \( S = (S(t))_{t \geq 0} \) be a stock price process with volatility \( \sigma \). State the price of a European call on \( S \) with maturity \( T \) and strike price \( K \).

(b) Suppose the value of one US dollar at time \( t \) equals \( \xi(t) \) Swedish crowns and that the price process \((\xi(t))_{0 \leq t \leq T}\) is a geometric Brownian motion with volatility \( \sigma \). Moreover, denote by \( r_f \) and \( r \) the US and Swedish interest rates, respectively.

Consider the right but not the obligation to buy one US dollar at the price \( K \) Swedish crowns at time \( T \). Use Part (a) to derive the price of this derivative at time \( t \) expressed in Swedish crowns.