1. (Black-Scholes model) Let $T > 1$ and $K > 0$ and consider a financial derivative of European type with payoff

$$Y = \left( \frac{S(T)}{S(T-1)} - K \right)^+$$

at time of maturity $T$. Find $\Pi_Y(t)$ if $0 \leq t \leq T - 1$.

Solution. Since

$$Y = \frac{1}{S(T-1)} (S(T) - KS(T-1))^+$$

the Black-Scholes call price formula yields

$$\Pi_Y(T-1) = \Phi\left( \frac{\ln \frac{1}{K} + \frac{r + \sigma^2}{2}}{\sigma} \right) - Ke^{-r} \Phi\left( \frac{\ln \frac{1}{K} + \frac{r - \sigma^2}{2}}{\sigma} \right).$$

Thus if $\tau = T - t$ and $0 \leq t \leq T - 1$,

$$\Pi_Y(t) = e^{-r(\tau-1)} \left\{ \Phi\left( \frac{\ln \frac{1}{K} + \frac{r + \sigma^2}{2}}{\sigma} \right) - Ke^{-r} \Phi\left( \frac{\ln \frac{1}{K} + \frac{r - \sigma^2}{2}}{\sigma} \right) \right\}.$$
2. (Binomial model in $T$ period with $d < r < u$) A financial derivative of European type pays the amount

\[ Y = \ln \frac{S(T)}{S(0)} \]

at time of maturity $T$. Find $\Pi_Y(0)$.

Solution. Using standard notation,

\[ Y = \ln \frac{S(T - 1)}{S(0)} + X_T \]

and, hence,

\[ \Pi_Y(T - 1) = e^{-r} \ln \frac{S(T - 1)}{S(0)} + e^{-r}(q_u u + q_d d) \]

where

\[ q_u = \frac{e^r - e^d}{e^u - e^r} = 1 - q_d. \]

In a similar way, if $T \geq 2$,

\[ \Pi_Y(T - 2) = e^{-r}(e^{-r} \ln \frac{S(T - 2)}{S(0)} + e^{-r}(q_u u + q_d d)) + e^{-2r}(q_u u + q_d d) \]

\[ = e^{-2r} \ln \frac{S(T - 2)}{S(0)} + 2e^{-2r}(q_u u + q_d d) \]

and by iteration

\[ \Pi_Y(0) = T e^{-Tr}(q_u u + q_d d). \]
3. A random variable $X$ has the density

$$f(x) = \begin{cases} \frac{1}{\pi} \sin^2 x & \text{if } |x| \leq \pi, \\ 0 & \text{otherwise.} \end{cases}$$

Find the characteristic function of $X$.

Solution. We have that

$$c_X(\xi) = E[e^{i\xi X}] = \int_{-\pi}^{\pi} e^{i\xi x} \frac{1}{\pi} \sin^2 x \, dx$$

$$= \int_{-\pi}^{\pi} (\cos \xi x + i \sin \xi x) \frac{1}{\pi} \sin^2 x \, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} \cos \xi x \sin^2 x \, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} (1 - \cos 2x) \cos \xi x \, dx$$

$$= \frac{1}{\pi} \int_{0}^{\pi} (1 - \cos 2x) \cos \xi x \, dx = \frac{1}{\pi} (\sin \pi \xi \frac{\sin \pi}{\xi} - a)$$

where

$$a = \int_{0}^{\pi} \cos 2x \cos \xi x \, dx$$

$$= \frac{1}{2} \int_{0}^{\pi} (\cos(2 + \xi)x + \cos(2 - \xi)x) \, dx$$

$$= \frac{1}{2} \left( \sin(2 + \xi)\pi - \sin(2 - \xi)\pi \right)$$

$$= \frac{2 \sin \pi \xi}{4 - \xi^2}.$$ 

Thus

$$c_X(\xi) = \frac{1}{\pi} \left( \sin \xi \frac{\sin \pi}{\xi} - \frac{2 \sin \pi \xi}{4 - \xi^2} \right).$$

4. (Black-Scholes model) Suppose $t < T$ and $\tau = T - t$. A simple financial derivative of European type with the payoff function $g \in \mathcal{P}$ has the price

$$\Pi_{g(S(T))}(t) = e^{-r\tau} E\left[ g(Se^{(r-\frac{\sigma^2}{2})\tau + \sigma \sqrt{\tau} \zeta}) \right]$$
at time $t$, where $s = S(t)$ is the stock price at time $t$ and $G \in N(0,1)$.

(a) A European call has the strike price $K$ and determination date $T$. Show that the call price at time $t$ equals $s\Phi(d_1) - Ke^{-rT}\Phi(d_2)$, where

$$d_1 = \frac{1}{\sigma \sqrt{T}} (\ln \frac{s}{K} + (r + \frac{\sigma^2}{2})T)$$

and $d_2 = d_1 - \sigma \sqrt{T}$.

(b) Show that the delta of the call in Part (a) equals $\Phi(d_1)$.

5. (Black-Scholes model) Suppose the value of one US dollar at time $t$ equals $\xi(t)$ Swedish crowns and that the price process $(\xi(t))_{0 \leq t \leq T}$ is a geometric Brownian motion with volatility $\sigma$. Moreover, denote by $r_f$ and $r$ the US and Swedish interest rates, respectively.

Consider the right but not the obligation to buy one US dollar at the price $K$ Swedish crowns at time $T$. Derive the price in Swedish crowns of this derivative at time $t$. 