SOLUTIONS
OPTIONS AND MATHEMATICS
(CTH[mve095], GU[MMA700])

No aids.
Each problem is worth 3 points.
Examiner: Christer Borell, telephone number 0705292322

1. (Black-Scholes model) Suppose $a$ and $b$ are positive constants. A derivative of European type pays the amount $Y = aS(T) + \frac{b}{S(T)}$ at time of maturity $T$.
   (a) Compute the time $t$ price of the derivative. (b) Compute the time $t$ delta of the derivative.

Solution. (a) Set $s = S(t)$. By the weak dominance principle in the Black-Scholes model $\Pi_{S(T)}(t) = s$ and, furthermore, if $\tau = T - t$,

$$\Pi_{S(T)}(t) = e^{-\tau r} E \left[ \frac{1}{s e^{(r - \frac{\sigma^2}{2})\tau} + \sigma \sqrt{\tau} G} \right]$$

where $G \in N(0, 1)$. Hence

$$\Pi_{S(T)}(t) = e^{-\tau r} (se^{(r - \frac{\sigma^2}{2})\tau})^{-1} E \left[ e^{-\sigma \sqrt{\tau} G} \right] =$$

$$\frac{1}{s} e^{(-2r + \frac{\sigma^2}{2})\tau} e^{\frac{\sigma^2}{2} \tau} = \frac{1}{s} e^{(\sigma^2 - 2r)\tau}$$

and we get

$$\Pi_Y(t) = as + \frac{b}{s} e^{(\sigma^2 - 2r)\tau}.$$  

(b) If $v(t, s) = \Pi_Y(t) = as + \frac{b}{s} e^{(\sigma^2 - 2r)\tau}$, then

$$\Delta(t) = v'_s(t, S(t)) = a - \frac{b}{S^2(t)} e^{(\sigma^2 - 2r)\tau}.$$  

2. (In this problem give only answers; please, do not hand in any solutions!) Let $W$ be a standard Brownian motion and set $U = W^2(1)$ and $V =$
\[ W(1)W(2) + W(3). \] Find (a) \( E[U] \) (b) \( E[V] \) (c) \( E[U^2] \) (d) \( E[V^2] \) (e) \( E[UV] \) (f) \( \text{Cov}(U, V) \) and (g) \( \text{Cor}(U, V) \).

Solution. Set \( X = W(1) \), \( Y = W(2) - W(1) \), and \( Z = W(3) - W(2) \). Then \( X, Y, \) and \( Z \) are independent, \( X, Y, Z \in N(0, 1) \), and

\[ U = X^2 \]

and

\[ V = X^2 + XY + X + Y + Z. \]

(a) \[ E[U] = E[X^2] = 1 \]


(c) \[ E[U^2] = E[X^4] = 3 \]

(d) \[ E[V^2] = E[(X(X + Y) + (X + Y + Z))^2] = E[X^2(X^2 + 2XY + Y^2)] + 2E[(X^2 + XY)(X + Y + Z)] + E[(X + Y + Z)^2] = (E[X^4] + E[X^2Y^2]) + 0 + \text{Var}(X + Y + Z) = 3 + 1 + 1 = 7 \]

Alternative solution:


(f)
Cov(U, V) = 2

\[ \text{Cor}(U, V) = \frac{2}{\sqrt{2} \sqrt{6}} = \frac{1}{\sqrt{3}} \]

3. (Black-Scholes model) Suppose \( 0 < a < b \) and \( 0 \leq t < T \). A financial derivative of European type pays the amount \( Y \) at time of maturity \( T \), where

\[ Y = \begin{cases} 
1 & \text{if } S(T) \in ]a, b], \\
0 & \text{if } S(T) \notin ]a, b[. 
\end{cases} \]

(a) Find \( \Pi_Y(t) \). (b) For which value on \( S(t) \) is \( \Pi_Y(t) \) maximal.

Solution. (a) Let \( H(x) = \)

\[ H_0(x) = \begin{cases} 
1 & \text{if } x > 0, \\
0 & \text{if } x \leq 0 
\end{cases} \quad \text{and} \quad H_1(x) = \begin{cases} 
1 & \text{if } x \geq 0, \\
0 & \text{if } x < 0. 
\end{cases} \]

Then

\[ Y = H_0(S(T) - a) - H_1(S(T) - b). \]

Moreover, if \( s = S(0) \) and \( \tau = T - t \),

\[ \Pi_{H_0(S(T)-a)}(t) = e^{-r\tau} \int_{-\infty}^{\infty} H_0(se^{(r-\frac{\sigma^2}{2})\tau-\sigma\sqrt{\tau}x} - a)\varphi(x)dx = \\
e^{-r\tau} \int_{-\infty}^{\frac{1}{\sigma\sqrt{\tau}}(\ln \frac{s}{a} + (r-\frac{\sigma^2}{2})\tau)} \varphi(x)dx = e^{-r\tau} \Phi\left( \frac{1}{\sigma\sqrt{\tau}}(\ln \frac{s}{a} + (r-\frac{\sigma^2}{2})\tau) \right) \]

and, in a similar way,

\[ \Pi_{H_1(S(T)-b)}(t) = e^{-r\tau} \Phi\left( \frac{1}{\sigma\sqrt{\tau}}(\ln \frac{s}{b} + (r-\frac{\sigma^2}{2})\tau) \right). \]

Thus

\[ \Pi_Y(t) = e^{-r\tau} \left( \Phi\left( \frac{1}{\sigma\sqrt{\tau}}(\ln \frac{s}{a} + (r-\frac{\sigma^2}{2})\tau) \right) - \Phi\left( \frac{1}{\sigma\sqrt{\tau}}(\ln \frac{s}{b} + (r-\frac{\sigma^2}{2})\tau) \right) \right). \]
(b) Set \( \Pi_Y(t) = v(s) \). Since \( s_a > s_b \) and \( \Phi \) is strictly increasing, it is obvious that \( v \) is a positive function. Moreover, \( v \) is continuous and

\[
\lim_{s \to \infty} v(s) = \lim_{s \to 0^+} v(s) = 0.
\]

From this we conclude that \( v \) attains a maximum and the derivative of \( v(s) \) vanishes at this point.

We have

\[
v'(s) = \frac{e^{-\tau r}}{s \sigma \sqrt{\pi}} \left( \varphi \left( \frac{1}{\sigma \sqrt{\tau}} (\ln \frac{s}{a} + (r - \frac{\sigma^2}{2}) \tau) \right) - \varphi \left( \frac{1}{\sigma \sqrt{\tau}} (\ln \frac{s}{b} + (r - \frac{\sigma^2}{2}) \tau) \right) \right)
\]

and, hence \( v'(s) = 0 \) if and only if

\[
(\ln \frac{s}{a} + (r - \frac{\sigma^2}{2}) \tau)^2 = (\ln \frac{s}{b} + (r - \frac{\sigma^2}{2}) \tau)^2.
\]

Thus

\[
\ln \frac{s}{a} + (r - \frac{\sigma^2}{2}) \tau = \pm (\ln \frac{s}{b} + (r - \frac{\sigma^2}{2}) \tau).
\]

Here the plus sign leads to \( a = b \), which is a contradiction, and we must have

\[
2 \ln s = \ln ab - 2(r - \frac{\sigma^2}{2}) \tau
\]

or

\[
s = \sqrt{abe^{-(r - \frac{\sigma^2}{2}) \tau}}.
\]

4. (Single-period binomial model and \( d < r < u \)) Let \( g : \{ S(0)e^{u}, S(0)e^{d} \} \to \mathbb{R} \) be a given function and suppose a derivative of European type pays the amount \( Y = g(S(1)) \) at time 1. Find a portfolio \( h = (h_S, h_B) \) which replicates the derivative.

5. Suppose \( \alpha \in \mathbb{R}, \sigma > 0 \) and let

\[
S(t) = S(0)e^{\alpha t + \sigma W(t)}, \quad t \geq 0
\]
be a geometric Brownian motion. Moreover, suppose $0 < t_1 < \ldots < t_n$ and $a_1 < b_1, \ldots, a_n < b_n$. Prove that

$$P[a_1 < S(t_1) < b_1, \ldots, a_n < S(t_n) < b_n]$$

$$= \int \cdots \int_{A_1 \times \ldots \times A_n} \left\{ \frac{1}{\sqrt{2\pi(t_k - t_{k-1})}} e^{-\frac{(x_k - x_{k-1})^2}{2(t_k - t_{k-1})}} \right\} dx_1 \ldots dx_n$$

where $x_0 = 0$, $t_0 = 0$, and

$$A_k = \left[ \frac{1}{\sigma} \left( \ln \frac{a_k}{S(0)} - \alpha t_k \right), \frac{1}{\sigma} \left( \ln \frac{b_k}{S(0)} - \alpha t_k \right) \right], \quad k = 1, \ldots, n.$$