1. (Binomial model with 2 periods and $u > r > d$) A European derivative pays the amount $Y$ at time of maturity $T = 2$, where

$$Y = \begin{cases} 
0, & \text{if } X_1 = X_2 \\
1, & \text{otherwise.}
\end{cases}$$

(a) Find the price $\Pi_Y(0)$ of the derivative at time zero. (b) Suppose $(h_S(t), h_B(t))_{t=0}^T$ is a self-financing portfolio which replicates the derivative. Find $h_S(0)$.

Solution. (a) Set $v(t) = \Pi_Y(t)$ and

$$q_u = \frac{e^r - e^d}{e^u - e^r} = 1 - q_d.$$ 

Then

$$v(2)|_{X_1 = u, X_2 = u} = 0$$
$$v(2)|_{X_1 = u, X_2 = d} = 1$$
$$v(2)|_{X_1 = d, X_2 = u} = 1$$
$$v(2)|_{X_1 = u, X_2 = u} = 0$$

and

$$v(1)|_{X_1 = u} = e^{-r}(q_u \cdot 0 + q_d \cdot 1) = e^{-r} q_d$$
$$v(1)|_{X_1 = d} = e^{-r}(q_u \cdot 1 + q_d \cdot 0) = e^{-r} q_u.$$ 

Hence

$$\Pi_Y(0) = v(0) = e^{-r}(q_u e^{-r} q_d + q_d e^{-r} q_u) = 2e^{-2r}q_u q_d.$$ 

(b) We have that $h_S(0) = h_S(1)$ and $h_B(0) = h_B(1)$. Hence

$$h_S(0)S(0)e^u + h_B(0)B(0)e^r = v(1)|_{X_1 = u}$$
$$h_S(0)S(0)e^d + h_B(0)B(0)e^r = v(1)|_{X_1 = d}$$
2

or

\[
\begin{align*}
    h_S(0)S(0)e^u + h_B(0)B(0)e^r &= e^{-r}q_d \\
    h_S(0)S(0)e^d + h_B(0)B(0)e^r &= e^{-r}q_u
\end{align*}
\]

and it follows that

\[
h_S(0) = e^{-r} \frac{q_d - q_u}{S(0)(e^u - e^d)}.
\]

2. (In this problem give only answers.) Let \( Z(t) = (Z_1(t), Z_2(t)), t \geq 0, \) be a standard Brownian motion in the plane and suppose \( T > 0 \). Set \( U = e^{2Z_1(T)} \) and \( V = e^{Z_1(T) + Z_2(2T)} \).

(a) Find \( E[U], E[V], \text{Var}(U), \text{Var}(V), \) and \( \text{Cov}(U, V) \).

(b) Find an \( a \in \mathbb{R} \) such that \( \text{Var}(U - aV) \leq \text{Var}(U - xV) \) for every \( x \in \mathbb{R} \).

Solution (to help the understanding of the answers). (a) In the following we will use that

\[
a_1Z_1(t_1) + a_2Z_2(t_2) \in N(0, a_1^2t_1 + a_2^2t_2)
\]

for all \( a_1, a_2 \in \mathbb{R} \) and \( t_1, t_2 \geq 0 \). Hence, if \( G \in N(0, 1) \),

\[
E[U] = E\left[e^{2\sqrt{T}G}\right] = e^{2T},
\]

\[
E[V] = E\left[e^{3\sqrt{T}G}\right] = e^{3T},
\]

\[
\text{Var}(U) = E[U^2] - (E[U])^2 = E\left[e^{4\sqrt{T}G}\right] - e^{4T} = e^{8T} - e^{4T},
\]

\[
\text{Var}(V) = E[V^2] - (E[V])^2 = E\left[e^{6\sqrt{T}G}\right] - e^{6T} = e^{12T} - e^{6T},
\]

\[
\text{Cov}(U, V) = E[UV] - E[U]E[V] = E\left[e^{\sqrt{T}G}\right] - e^{2T}e^{\frac{3}{2}T} = e^{\frac{11}{2}T} - e^{\frac{7}{2}T}.
\]

(b) Set \( U_0 = U - E[U] \) and \( V_0 = V - E[V] \). We have

\[
f(x) = \text{def} \ \text{Var}(U - xV) = E[(U_0 - xV_0)^2]
\]

\[
= E[U_0^2] - 2xE[U_0V_0] + x^2E[V_0^2]
\]

\[
= (x\sqrt{E[V_0^2]} - \frac{E[U_0V_0]}{\sqrt{E[V_0^2]}})^2 + E[U_0^2] - (\frac{E[U_0V_0]}{\sqrt{E[V_0^2]}})^2.
\]

Hence

\[
\min f = f(a)
\]
where
\[ a = \frac{\text{Cov}(U, V)}{\text{Var}(V)} = \frac{e^{\frac{11}{2}T} - e^{\frac{7}{2}T}}{e^{6T} - e^{3T}} \]
\[ = \frac{e^{\frac{5}{2}T} - e^{\frac{3}{2}T}}{e^{3T} - 1} = \frac{e^{\frac{1}{2}T}(e^{T} + 1)}{e^{2T} + e^{T} + 1} \]

3. (Black-Scholes model) Suppose $0 < t_0 < T$ and $K > 0$. A financial derivative of European type pays the amount $Y = \left( \frac{S(T)}{S(t_0)} - K \right)^+$ at time of maturity $T$. Find the delta of the option at time $t$ if (a) $0 < t < t_0$ (b) $t_0 < t < T$.

Solution. We first solve Part (b). Note that
\[ Y = \frac{1}{S(t_0)}(S(T) - KS(t_0))^+ \]
and, accordingly from this, if $t_0 \leq t < T$,
\[ \Pi_Y(t) = \frac{1}{S(t_0)} c(t, S(t), KS(t_0), T) \]
\[ = \frac{1}{S(t_0)} \{ S(t)\Phi(d_1(t)) - KS(t_0)e^{-r(T-t)}\Phi(d_2(t)) \} \]
where
\[ d_1(t) = \frac{\ln \frac{S(t)}{KS(t_0)} + (r + \frac{\sigma^2}{2})(T - t)}{\sigma\sqrt{T-t}} \]
and
\[ d_2(t) = \frac{\ln \frac{S(t)}{KS(t_0)} + (r - \frac{\sigma^2}{2})(T - t)}{\sigma\sqrt{T-t}}. \]
In particular,
\[ \Pi_Y(t_0) = \Phi\left( -\ln K + (r + \frac{\sigma^2}{2})(T - t_0) \frac{1}{\sigma\sqrt{T-t_0}} \right) - Ke^{-r(T-t_0)}\Phi\left( -\ln K + (r - \frac{\sigma^2}{2})(T - t_0) \frac{1}{\sigma\sqrt{T-t_0}} \right) \]
and, moreover, from the known delta of a European call we get
\[ \Delta(t) = \frac{1}{S(t_0)} \Phi\left( \frac{\ln \frac{S(t)}{KS(t_0)} + (r + \frac{\sigma^2}{2})(T - t)}{\sigma\sqrt{T-t}} \right), \text{ if } t_0 < t < T. \]
We next treat Part (a). If $s = S(t)$ and $0 < t < t_0$,

$$\Pi_Y(t) = e^{-r(t_0-t)}\Pi_Y(t_0)$$

since $\Pi_Y(t_0)$ is known at time $t$. Moreover, $\Pi_Y(t)$ is independent of $s$ and we have

$$\Delta(t) = 0, \text{ if } 0 < t < t_0.$$

4. (Dominance principle) State and prove the Put-Call Parity Theorem.

5. (Black-Scholes model) Consider a European call option on $S$ with strike price $K$ and time of maturity $T$. Prove that the delta of the call at time $t < T$ equals

$$\Phi\left(\frac{\ln S(t) - \ln K + (r + \frac{\sigma^2}{2})(T - t)}{\sigma \sqrt{T - t}}\right).$$