1. (The one period binomial model, where $0 < d < r < u$) Consider a call with the payoff $Y = \frac{1}{2} \left| \frac{S(1)}{S(0)} - \frac{S(0)}{S(1)} \right|$ at the termination date 1. Find the replicating strategy of the derivative at time 0.

Solution: Let $S(0) = s$ and $S(1) = se^X$, where $X = u$ or $d$. If $(h_S, h_B)$ denotes the replicating strategy at time 0 we have

$$h_Sse^u + h_BB(0)e^r = \sinh u$$

and

$$h_Sse^d + h_BB(0)e^r = \sinh d.$$ 

From this it follows that

$$h_S(e^u - e^d) = \sinh u - \sinh d$$

and

$$h_S = \frac{\sinh u - \sinh d}{s(e^u - e^d)}.$$ 

Moreover, we get

$$h_BB(0)(e^{r+u} - e^{r+d}) = e^u \sinh d - e^d \sinh u$$

and

$$h_B = \frac{e^u \sinh d - e^d \sinh u}{B(0)e^r(e^u - e^d)}.$$ 

ANSWER: $\frac{\sinh u - \sinh d}{S(0)(e^u - e^d)}$ (or $= \frac{1}{2S(0)}(1+e^{-u-d})$) units of the stock and $\frac{e^u \sinh d - e^d \sinh u}{B(0)e^r(e^u - e^d)}$ (or $= -\frac{e^{-r}}{2B(0)}(e^{-u} + e^{-d})$) units of the bond.
2. (Black-Scholes model) Suppose $t^*, T_0, T$, and $\delta$ are positive numbers satisfying the inequalities $T_0 < t^* < T$ and $\delta < 1$. Moreover, suppose $t < t^*$. A stock pays the dividend $\delta S(t^*)$ at time $t^*$. Find the price $\Pi_Y(t)$ at time $t$ of a derivative of European type paying the amount

$$Y = \left( \frac{S(T)}{S(T_0)} - 1 \right)^+$$

at time of maturity $T$.

Solution. Set $s = S(t)$ and $\tau = T - t$.

To begin with we assume $T_0 \leq t < t^*$. If

$$g(x) = \left( \frac{x}{S(T_0)} - 1 \right)^+ = \frac{1}{S(T_0)} (x - S(T_0))^+$$

we know that

$$\Pi_Y(t) = e^{-\tau r} E \left[ g((1 - \delta) s e^{(r - \frac{\sigma^2}{2}) \tau + \sigma \sqrt{\tau} G}) \right]$$

where $G \in N(0, 1)$. Hence, by the Black-Scholes price formula for a European call,

$$\Pi_Y(t) = \frac{1}{S(T_0)} c(t, (1 - \delta)s, S(T_0), T)$$

$$= \frac{1}{S(T_0)} \left\{ (1 - \delta) S(t) \Phi(D_1(t)) - S(T_0) e^{-\tau r} \Phi((D_2(t)) \right\}$$

where

$$D_1(t) = \frac{1}{\sigma \sqrt{\tau}} \left( \ln \left( \frac{1 - \delta}{S(T_0)} \right) + (r + \frac{\sigma^2}{2}) \tau \right)$$

and

$$D_2(t) = \frac{1}{\sigma \sqrt{\tau}} \left( \ln \left( \frac{1 - \delta}{S(T_0)} \right) + (r - \frac{\sigma^2}{2}) \tau \right).$$

In particular,

$$\Pi_Y(T_0) = (1 - \delta) \Phi(D_1(T_0)) - e^{-\tau(T - T_0)} \Phi((D_2(T_0))$$

where

$$D_1(T_0) = \frac{1}{\sigma \sqrt{T - T_0}} \left( \ln(1 - \delta) + (r + \frac{\sigma^2}{2}) (T - T_0) \right)$$
and
\[ D_2(T_0) = \frac{1}{\sigma \sqrt{T - T_0}} (\ln(1 - \delta) + (r - \frac{\sigma^2}{2})(T - T_0)). \]

Since \( \Pi_Y(T_0) \) is non-random (= a numerical constant) we conclude that
\[ \Pi_Y(t) = e^{-r(T_0 - t)} \{(1 - \delta)\Phi(D_1(T_0)) - e^{-r(T_0 - T_0)}\Phi((D_2(T_0)))\} \text{ if } t < T_0. \]

3. (Black-Scholes model) Let \( a, K, T > 0 \). A financial derivative of European type pays the amount \( Y = (\min(S(T) - K, a))^+ \) at time of maturity \( T \). Show that the delta of the derivative is positive and does not exceed
\[ \frac{\ln(1 + \frac{a}{K})}{\sigma \sqrt{2\pi(T - t)}} \]

at time \( t < T \).

Solution. Note that \( Y = (S(T) - K)^+ - (S(T) - (a + K))^+ \). The delta of a call is standard (see Problem 4) and we get that the delta of \( Y \) at time \( t \) equals
\[ \Delta_Y(t) = \Phi\left( \frac{1}{\sigma \sqrt{\tau}} \left( (\ln \frac{S(t)}{K} + (r + \frac{\sigma^2}{2})\tau) - (\ln \frac{S(t)}{a + K} + (r + \frac{\sigma^2}{2})\tau) \right) \right) \]

where \( \tau = T - t \). Hence \( \Delta_Y(t) > 0 \) since \( \Phi \) and \( \ln \) are increasing in the strict sense. Moreover, if
\[ \varphi(u) = \frac{1}{\sqrt{2\pi}} \exp(-u^2/2) \]

we have
\[ \Delta_Y(t) = \left\{ \frac{1}{\sigma \sqrt{\tau}} \left( (\ln \frac{S(t)}{K} + (r + \frac{\sigma^2}{2})\tau) - (\ln \frac{S(t)}{a + K} + (r + \frac{\sigma^2}{2})\tau) \right) \right\} \varphi(\xi). \]

for an appropriate \( \xi \in \left] \frac{1}{\sqrt{2\pi}} (\ln \frac{S(t)}{a + K} + (r + \frac{\sigma^2}{2})\tau), \frac{1}{\sigma \sqrt{\tau}} (\ln \frac{S(t)}{K} + (r + \frac{\sigma^2}{2})\tau) \right[. \]

But \( \varphi(\xi) \leq \frac{1}{\sqrt{2\pi}} \) and we get
\[ \Delta_Y(t) \leq \frac{1}{\sigma \sqrt{\tau}} (\ln (a + K) - \ln K) \frac{1}{\sqrt{2\pi}}. \]
and the result is immediate.

4. (Black-Scholes model) Suppose \( t < T \) and \( \tau = T - t \). A simple financial derivative of European type with the payoff function \( g \in \mathcal{P} \) has the price

\[
\Pi_{g(S(T))}(t) = e^{-r\tau} E \left[ g(s e^{(r-\frac{\sigma^2}{2})\tau + \sigma \sqrt{\tau} G}) \right]
\]

at time \( t \), where \( s = S(t) \) is the stock price at time \( t \) and \( G \in N(0,1) \).

(a) A European call has the strike price \( K \) and determination date \( T \). Show that the call price equals \( s \Phi(d_1) - K e^{-r\tau} \Phi(d_2) \), where

\[
d_1 = \frac{1}{\sigma \sqrt{\tau}} (\ln \frac{s}{K} + (r + \frac{\sigma^2}{2})\tau)
\]

and \( d_2 = d_1 - \sigma \sqrt{\tau} \).

(b) Show that the delta of the call in Part (a) equals \( \Phi(d_1) \).

5. (Black-Scholes model) A European call on a US dollar has the strike strike price \( K \) and determination date \( T \). Derive the price of the derivative at time \( t \), if the US interest rate equals \( r_f \) and the volatility of the exchange rate process, quoted as crowns per dollar, equals \( \sigma \). As usual the Swedish interest rate is denoted by \( r \).