SOLUTIONS
OPTIONS AND MATHEMATICS
(CTH[mve095], GU[MMA700])

January 17, 2009, morning (4 hours), v
No aids.
Examiner: Christer Borell, telephone number 0705292322
Each problem is worth 3 points.

1. (The one period binomial model, where $d < 0 < r < u$) Consider a put with the payoff $Y = (S(0) - S(1))^+$ at the termination date 1. Find the replicating strategy of the derivative at time 0.

Solution: Let $S(0) = s$ and $S(1) = se^X$, where $X = u$ or d. If (h_S, h_B) denotes the replicating strategy at time 0 we have

$$h_S se^u + h_B B(0)e^r = 0$$

and

$$h_S se^d + h_B B(0)e^r = s(1 - e^d).$$

From this it follows that

$$h_S s(e^u - e^d) = s(e^d - 1)$$

and

$$h_S = \frac{e^d - 1}{e^u - e^d}.$$

Moreover, we get

$$h_B = -\frac{1}{B(0)} h_S se^{u-r} = \frac{se^{u-r}}{B(0)} \frac{1 - e^d}{e^u - e^d}.$$

2. Suppose $\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ and $\Phi(x) = \int_{-\infty}^{x} \varphi(t)dt$, $-\infty < x < \infty$. Prove that

$$1 - \Phi(x) \leq \frac{\varphi(x)}{x}, \text{ if } x > 0,$$

and

$$1-\Phi(x) \geq \frac{x\varphi(x)}{1+x^2}, \text{ if } x \in \mathbf{R}.$$

Solution. For any $x > 0$,

$$1-\Phi(x) = \int_x^\infty \varphi(t)dt = \int_x^\infty \frac{1}{t} t\varphi(t)dt$$

$$\leq \int_x^\infty \frac{1}{x} t\varphi(t)dt = \frac{1}{x}\left[-\varphi(t)\right]_{t=x}^{t=\infty} = \frac{\varphi(x)}{x}.$$

This proves the first inequality. To prove the second inequality define

$$f(x) = (1+x^2)(1-\Phi(x)) - x\varphi(x), \text{ if } x \in \mathbf{R}.$$

It is obivous that $f(x) > 0$ if $x \leq 0$ and therefore it is enough to prove that $f(x) \geq 0$ for every $x > 0$. To this end, first note that

$$\lim_{x\to\infty}(1+x^2)(1-\Phi(x)) = 0$$

since $0 \leq 1-\Phi(x) \leq \frac{\varphi(x)}{x} = \frac{1}{x\sqrt{2\pi}}e^{-\frac{x^2}{2}}$ for every $x > 0$. Hence

$$\lim_{x\to\infty} f(x) = 0$$

and it is enough to show that $f'(x) \leq 0$ if $x > 0$. Now for every $x > 0$,

$$f'(x) = 2x(1-\Phi(x)) - (1+x^2)\varphi(x) - \varphi(x) + x^2\varphi(x)$$

$$= 2x(1-\Phi(x) - \frac{\varphi(x)}{x}) \leq 0$$

and we are done.

3. (Black-Scholes model) (a) Consider a derivative of European type with the payoff

$$Y = \frac{1}{n}\sum_{k=1}^{n} S(\frac{kT}{n})$$

at time of maturity T. Find $\Pi_Y(0)$.

(b) Consider a derivative of European type with the payoff

$$Z = \left\{ \prod_{k=1}^{n} S(\frac{kT}{n}) \right\}^{\frac{1}{n}}$$

at time of maturity T. Find $\Pi_Z(0)$. (Hint: $1^2 + 2^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$)

Solution. (a) Consider a derivative paying the amount $Y_k = S(\frac{kT}{n})$ at time T. Then

$$\Pi_Y(0) = \frac{1}{n} \sum_{k=1}^{n} \Pi_{Y_k}(0).$$

Moreover, $\Pi_{Y_k}(\frac{kT}{n}) = e^{-(T-\frac{kT}{n})r} S(\frac{kT}{n})$ and, hence,

$$\Pi_{Y_k}(0) = e^{-(T-\frac{kT}{n})r} S(0).$$

Thus

$$\Pi_Y(0) = \frac{S(0)}{n} \sum_{k=1}^{n} e^{-(1-\frac{k}{n})Tr}$$

$$= \frac{S(0)}{n} \sum_{i=0}^{n-1} e^{-iTr/n} = \frac{S(0)}{n} \frac{1-e^{-Tr}}{1-e^{-Tr/n}}.$$

(b) If $S(0) = s$,

$$\Pi_Z(0) = e^{-rT} E \left[\left\{ \prod_{k=1}^{n} s e^{(r-\frac{\sigma^2}{2})\frac{kT}{n} + \sigma W(\frac{kT}{n})} \right\}^{\frac{1}{n}} \right]$$

$$= s e^{-rT + (r-\frac{\sigma^2}{2})\frac{(n+1)T}{2n}} E \left[e^{\frac{\sigma}{n} \sum_{k=1}^{n} W(\frac{kT}{n})} \right].$$

Set $V_i = W(\frac{iT}{n})$, $i = 0, ..., n$. Then

$$\sum_{k=1}^{n} W(\frac{kT}{n}) = V_1 + ... + V_n$$

$$= V_1 + ... + V_{n-2} + 2V_{n-1} + (V_n - V_{n-1})$$

$$= V_1 + ... + V_{n-3} + 3V_{n-2} + 2(V_{n-1} - V_{n-2}) + (V_n - V_{n-1})$$
$$= n(V_1 - V_0) + ... + 2(V_{n-1} - V_{n-2}) + (V_n - V_{n-1})$$

and we get

$$E\left[e^{\frac{\sigma}{n}\sum_{k=1}^{n} W(\frac{kT}{n})}\right] = \prod_{k=1}^{n} E\left[e^{\frac{\sigma(n+1-k)}{n}(V_k - V_{k-1})}\right] = e^{\frac{\sigma^2}{2n^2}(n^2+...+2^2+1^2)\frac{T}{n}}$$

$$= e^{\frac{\sigma^2}{2n^2}\frac{n(n+1)(2n+1)}{6}\frac{T}{n}} = e^{\sigma^2 T\frac{(n+1)(2n+1)}{12n^2}}.$$

Thus

$$\Pi_Z(0) = se^{-rT+(r-\frac{\sigma^2}{2})\frac{(n+1)T}{2n}+\sigma^2 T\frac{(n+1)(2n+1)}{12n^2}} = S(0)e^{(\frac{1-n}{2n}r+\frac{1-n^2}{12n^2}\sigma^2)T}.$$

4. Let $(X_n)_{n=1}^{\infty}$ be an i.i.d. such that $P\left[X_1 = 1\right] = P\left[X_1 = -1\right] = \frac{1}{2}$ and set

$$Y_n = \frac{1}{\sqrt{n}}(X_1 + ... + X_n),\ n \in \mathbf{N}_+.$$

Prove that $Y_n \to G$, where $G \in N(0,1)$.

5. (Black-Scholes model) Suppose $\tau = T - t > 0$ and

$$d_1 = \frac{1}{\sigma\sqrt{\tau}}(\ln\frac{s}{K} + (r + \frac{\sigma^2}{2})\tau).$$

Prove that

$$\frac{\partial c}{\partial s}(t, s, K, T) = \Phi(d_1).$$

(Hint: $c(t, s, K, T) = s\Phi(d_1) - Ke^{-r\tau}\Phi(d_2)$, where $d_2 = d_1 - \sigma\sqrt{\tau}$)