1. (Black-Scholes model) A derivative of European type pays the amount $Y = \frac{S(T)}{S(T/2)}$ at time of maturity $T$. Find $\Pi_Y(0)$.

Solution. For any $t \in [0, T]$ and real number $a$, $\Pi_{aS(T)}(t) = aS(t)$ and, hence,

$$\Pi_Y(T/2) = \Pi_{\frac{1}{S(T/2)} S(T)}(T/2) = \frac{1}{S(T/2)} \Pi_{S(T)}(T/2)$$

$$= \frac{1}{S(T/2)} S(T/2) = 1.$$

Accordingly from this,

$$\Pi_Y(0) = e^{-\frac{rT}{2}}.$$

2. Suppose $Z = (Z_1(t), Z_2(t))_{t \geq 0}$ is a standard Brownian motion in the plane. Find $E \left[ \sqrt{Z_1^2(t) + Z_2^2(t)} \right]$ if $t \geq 0$.

Solution. Let $t \geq 0$ be fixed. Since $(Z_1(t), Z_2(t))$ has the same distribution as $\sqrt{t}(Z_1(1), Z_2(1))$,

$$E \left[ \sqrt{Z_1^2(t) + Z_2^2(t)} \right] = E \left[ \sqrt{t(Z_1^2(1) + Z_2^2(1))} \right]$$

$$= \sqrt{t} \int_{\mathbb{R}^2} \int \sqrt{x^2 + y^2} e^{-\frac{x^2+y^2}{2}} \frac{dxdy}{2\pi} = \left[ \text{polar coordinates} \right]$$
\[
\begin{aligned}
&= \sqrt{t} \int_0^\infty \int_0^{2\pi} r^2 e^{-r^2 \frac{1}{\pi}} dr d\theta = \sqrt{t} \int_0^\infty r^2 e^{-r^2} dr = \left[ \text{partial integration} \right] \\
&= \sqrt{t} \int_0^\infty e^{-r^2} dr = \sqrt{\frac{\pi t}{2}}.
\end{aligned}
\]

3. (Black-Scholes model) Suppose $K$ is a positive real number and consider a simple derivative of European type with the payoff

\[ Y = \left( \frac{1}{S(T)} - K \right)^+ \]

at time of maturity $T$. Moreover, suppose $0 < t^* < T$ and $0 < \delta < 1$. Find $\Pi_Y(0)$ if the stock pays the dividend $\delta S(t^*)$ at time $t^*$.

Solution. Let $s = S(0)$ and suppose $G \in N(0, 1)$. We have

\[ \Pi_Y(0) = e^{-rT} E \left[ \frac{1}{(1-\delta)se^{(r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}G}} - K \right]^+ \]

\[ = \frac{e^{-rT}}{(1-\delta)s} E \left[ (e^{-(r-\frac{\sigma^2}{2})T-\sigma\sqrt{T}G} - L)^+ \right] \]

where $L = (1-\delta)sK$. Here

\[ E \left[ (e^{-(r-\frac{\sigma^2}{2})T-\sigma\sqrt{T}G} - L)^+ \right] = \int_{-\infty}^{-\frac{1}{\sigma\sqrt{T}}(\ln L + (r-\frac{\sigma^2}{2})T)} e^{-(r-\frac{\sigma^2}{2})T-\sigma\sqrt{T}x-L}e^{-x^2} \frac{dx}{\sqrt{2\pi}} \]

\[ = e^{(r^2-r)T} \int_{-\infty}^{-\frac{1}{\sigma\sqrt{T}}(\ln L + (r-\frac{\sigma^2}{2})T)} e^{-\frac{1}{2}(x+\sigma\sqrt{T})^2} \frac{dx}{\sqrt{2\pi}} - L\Phi(-\frac{1}{\sigma\sqrt{T}}(\ln L + (r-\frac{\sigma^2}{2})T)) \]

\[ = e^{(r^2-r)T} \Phi\left( -\frac{1}{\sigma\sqrt{T}}(\ln L + (r-\frac{3}{2}\sigma^2)T) \right) - L\Phi\left( -\frac{1}{\sigma\sqrt{T}}(\ln L + (r-\frac{\sigma^2}{2})T) \right). \]

Thus

\[ \Pi_Y(0) = \frac{e^{(r^2-2r)T}}{(1-\delta)s} \Phi\left( -\frac{1}{\sigma\sqrt{T}}(\ln L + (r-\frac{3}{2}\sigma^2)T) \right) - e^{-rT}K\Phi\left( -\frac{1}{\sigma\sqrt{T}}(\ln L + (r-\frac{\sigma^2}{2})T) \right). \]
4. Prove that there exists an arbitrage portfolio in the single-period binomial model if and only if
\[ r \notin ]d, u[. \]

5. (Black-Scholes model) Consider a European call on a stock with price process \((S(t))_{t \geq 0}\). If \(K\) denotes strike price and \(T\) time of maturity, the Black-Scholes price of the call at time \(t < T\) equals
\[ c(t, S(t), K, T)) = s\Phi(d_1) - Ke^{-r\tau}\Phi(d_2), \]
where \(\tau = T - t\) and
\[ d_1 = d_2 + \sigma \sqrt{\tau} = \frac{1}{\sigma \sqrt{\tau}}(\ln \frac{S(t)}{K} + (r + \frac{\sigma^2}{2})\tau). \]
(a) Find the delta of the call.
(b) How is the call price formula changed if the stock price pays the dividend \(D\) at time \(t^* \in ]t, T[\), where \(D\) is a fixed amount known at time \(t\)?