1. (Binomial model) Suppose \( T = 3, u > r > 0, \) and \( d = -u. \) A derivative of European type has the payoff \( Y \) at time of maturity \( T, \) where

\[
Y = \begin{cases} 
1, & \text{if } X_1 = X_2 = X_3, \\
0, & \text{otherwise.}
\end{cases}
\]

Find \( v(0) \) (the answer may contain the martingale probabilities \( q_u \) and \( q_d, \) which must, however, be defined explicitly).

Solution. We have

\[
q_u = \frac{e^r - e^{-u}}{e^u - e^{-u}} \quad \text{and} \quad q_d = \frac{e^u - e^r}{e^u - e^{-u}}.
\]

Introducing \( v(t) = v(t), \) it follows that

\[
\begin{cases} 
v(2) |_{X_1 = u, X_2 = u} = e^{-r}(q_u \cdot 1 + q_d \cdot 0) = e^{-r}q_u \\
v(2) |_{X_1 = u, X_2 = d} = e^{-r}(q_u \cdot 0 + q_d \cdot 0) = 0 \\
v(2) |_{X_1 = d, X_2 = u} = e^{-r}(q_u \cdot 0 + q_d \cdot 0) = 0 \\
v(2) |_{X_1 = d, X_2 = d} = e^{-r}(q_u \cdot 0 + q_d \cdot 1) = e^{-r}q_d
\end{cases}
\]

and

\[
\begin{cases} 
v(1) |_{X_1 = u} = e^{-r}(q_u e^{-r}q_u + q_d \cdot 0) = e^{-2r}q_u^2 \\
v(1) |_{X_1 = d} = e^{-r}(q_u \cdot 0 + q_d e^{-r}q_d) = e^{-2r}q_d^2.
\end{cases}
\]

Thus

\[
v(0) = e^{-r}(q_u e^{-2r}q_u^2 + q_d e^{-2r}q_d^2) = e^{-3r}(q_u^3 + q_d^3).
\]
Alternative solution. We have $Y = 1_{\{S(0)e^{3u}, S(0)e^{-3u}\}}(S(3))$ and the derivative is simple. Hence

$$
\Pi_Y(0) = e^{-3r} \sum_{k=0}^{3} \binom{3}{k} q_u^k q_d^{3-k} 1_{\{S(0)e^{3u}, S(0)e^{-3u}\}}(S(0)e^{ku+(3-k)(-u)})
$$

$$
= e^{-3r} \sum_{k \in \{0,3\}} \binom{3}{k} q_u^k q_d^{3-k} = e^{-3r}(q_u^3 + q_d^3).
$$

2. Suppose $Z = (Z_1(t), Z_2(t))_{t \geq 0}$ is a standard Brownian motion in the plane and define $R(t) = |Z(t)| = \sqrt{Z_1^2(t) + Z_2^2(t)}$, $t \geq 0$. Find $E\left[e^{\xi R^2(t)}\right]$ if $t > 0$ and $\xi < \frac{1}{2t}$.

Solution. Suppose $t > 0$, $\xi < \frac{1}{2t}$, and $G \in N(0, 1)$. Then

$$
E\left[e^{\xi R^2(t)}\right] = E\left[e^{\xi Z_1^2(t)}e^{\xi Z_2^2(t)}\right] = E\left[e^{\xi Z_1^2(t)}\right] E\left[e^{\xi Z_2^2(t)}\right]
$$

$$
= \left(E\left[e^{\xi G^2}\right]\right)^2
$$

and setting $\eta = \xi t$,

$$
E\left[e^{\gamma G^2}\right] = \int_{-\infty}^{\infty} e^{\eta x^2} e^{-x^2/2} dx = \sqrt{2\pi} \int_{-\infty}^{\infty} e^{-x^2/2} dx = \sqrt{2\pi} \int_{-\infty}^{\infty} e^{-y^2/2} dy
$$

$$
= \frac{1}{\sqrt{1-2\eta}}.
$$

Hence,

$$
E\left[e^{\xi R^2(t)}\right] = \frac{1}{1-2\xi t}.
$$

3. (Black-Scholes model) A derivative of European type pays the amount

$$
Y = 1 + S(T) \ln S(T)
$$
at time of maturity $T$. (a) Find $\Pi_Y(t)$. (b) Find a hedging portfolio of the derivative at time $t$.

Solution. (a) If $s = S(t)$, $\tau = T - t$, and $G \in N(0, 1)$, then

$$\Pi_Y(t) = e^{-rt} E \left[ 1 + se^{(r - \frac{\sigma^2}{2})\tau + \sigma \sqrt{\tau} G} \left\{ \ln s + (r - \frac{\sigma^2}{2})\tau + \sigma \sqrt{\tau} G \right\} \right]$$

$$= e^{-rt} + s \left\{ \ln s + (r - \frac{\sigma^2}{2})\tau \right\} e^{\frac{\sigma^2}{2} \tau} E \left[ e^{\sigma \sqrt{\tau} G} \right] + s \sigma \sqrt{\tau} E \left[ Ge^{\frac{\sigma^2}{2} \tau + \sigma \sqrt{\tau} G} \right]$$

$$= e^{-rt} + s \left\{ \ln s + (r - \frac{\sigma^2}{2})\tau \right\} + s \sigma \sqrt{\tau} \int_{-\infty}^{\infty} xe^{-(x-\frac{\sigma^2}{2})^2} \frac{dx}{\sqrt{2\pi}}$$

$$= e^{-rt} + s \left\{ \ln s + (r - \frac{\sigma^2}{2})\tau \right\} + s \sigma \sqrt{\tau} \int_{-\infty}^{\infty} ye^{-\frac{y^2}{2}} \frac{dy}{\sqrt{2\pi}}$$

$$= e^{-rt} + s \left\{ \ln s + (r - \frac{\sigma^2}{2})\tau \right\} + s \sigma^2 \tau$$

$$= e^{-rt} + s \ln s + s(r + \frac{\sigma^2}{2})\tau$$

$$= e^{-rt} + S(t) \ln S(t) + S(t)(r + \frac{\sigma^2}{2})\tau$$

(b) A portfolio with

$$h_S(t) = \left( \frac{\partial}{\partial s} \left\{ e^{-rt} + s \ln s + s(r + \frac{\sigma^2}{2})\tau \right\} \right)_{s=S(t)}$$

$$= 1 + (r + \frac{\sigma^2}{2})\tau + \ln S(t)$$

units of the stock and

$$h_B(t) = \frac{e^{-rt} + S(t) \ln S(t) + S(t)(r + \frac{\sigma^2}{2})\tau - S(t)(1 + (r + \frac{\sigma^2}{2})\tau + \ln S(t)))}{B(t)}$$

$$= (e^{-rt} - S(t))/B(t)$$
units of the bond is a hedging portfolio at time $t$.

4. (Dominance Principle) State and prove the Put-Call Parity relation.

5. (Dominance Principle) Suppose $t_0 < t^* < T$ and let $D$ be a positive number, which is known at time $t_0$. Now consider an American put with strike $K$ and time of maturity $T$, where the underlying stock pays the dividend $D$ at time $t^*$ and

$$D \geq K(e^{r(t^*-t_0)} - 1).$$

Prove that it is not optimal to exercise the put in the time interval $]t_0, t^*[$.