SOLUTIONS
OPTIONS AND MATHEMATICS
(CTH[$mve095$], GU[$MMA700$])

May 24, 2008, morning (4 hours), v
No aids.
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Each problem is worth 3 points.

1. (Binomial model) Suppose $T = 3$, $u > r > 0$, and $d = -u$. A derivative of European type has the payoff Y at time of maturity T, where

$$Y = \begin{cases} 1, & \text{if } X_1 = X_2 = X_3, \\ 0, & \text{otherwise.} \end{cases}$$

Find $\Pi_Y(0)$ (the answer may contain the martingale probabilities q_u and q_d, which must, however, be defined explicitely).

Solution. We have

$$q_u = \frac{e^r - e^{-u}}{e^u - e^{-u}} \text{ and } q_d = \frac{e^u - e^r}{e^u - e^{-u}}.$$

Introducing $\Pi_Y(t) = v(t)$, it follows that

$$\begin{cases} v(2)_{|X_1=u,X_2=u} = e^{-r}(q_u \cdot 1 + q_d \cdot 0) = e^{-r} q_u \\ v(2)_{|X_1=u,X_2=d} = e^{-r}(q_u \cdot 0 + q_d \cdot 0) = 0 \\ v(2)_{|X_1=d,X_2=u} = e^{-r}(q_u \cdot 0 + q_d \cdot 0) = 0 \\ v(2)_{|X_1=d,X_2=d} = e^{-r}(q_u \cdot 0 + q_d \cdot 1) = e^{-r} q_d \end{cases}$$

and

$$\begin{cases} v(1)_{|X_1=u} = e^{-r}(q_u e^{-r} q_u + q_d \cdot 0) = e^{-2r} q_u^2 \\ v(1)_{|X_1=d} = e^{-r}(q_u \cdot 0 + q_d e^{-r} q_d) = e^{-2r} q_d^2. \end{cases}$$

Thus

$$v(0) = e^{-r}(q_u e^{-2r} q_u^2 + q_d e^{-2r} q_d^2) = e^{-3r}(q_u^3 + q_d^3).$$

Alternative solution. We have $Y = 1_{\{S(0)e^{3u}, S(0)e^{-3u}\}}(S(3))$ and the derivative is simple. Hence

$$\Pi_Y(0) = e^{-3r} \sum_{k=0}^{3} \left(\begin{array}{c} 3 \\ k \end{array} \right) q_u^k q_d^{3-k} 1_{\{S(0)e^{3u}, S(0)e^{-3u}\}}(S(0)e^{ku+(3-k)(-u)})$$

$$= e^{-3r} \sum_{k \in \{0,3\}} \left(\begin{array}{c} 3 \\ k \end{array} \right) q_u^k q_d^{3-k} = e^{-3r}(q_u^3 + q_d^3).$$

2. Suppose $Z = (Z_1(t), Z_2(t))_{t \geq 0}$ is a standard Brownian motion in the plane and define $R(t) =\mid Z(t) \mid= \sqrt{Z_1^2(t) + Z_2^2(t)}$, $t \geq 0$. Find $E\left[e^{\xi R^2(t)}\right]$ if $t > 0$ and $\xi < \frac{1}{2t}$.

Solution. Suppose $t > 0$, $\xi < \frac{1}{2t}$, and $G \in N(0,1)$. Then

$$E\left[e^{\xi R^2(t)}\right] = E\left[e^{\xi Z_1^2(t)} e^{\xi Z_2^2(t)}\right] = E\left[e^{\xi Z_1^2(t)}\right] E\left[e^{\xi Z_2^2(t)}\right]$$

$$= (E\left[e^{\xi t G^2}\right])^2$$

and setting $\eta = \xi t$,

$$E\left[e^{\eta G^2}\right] = \int_{-\infty}^{\infty} e^{\eta x^2} e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}}$$

$$= \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}(1-2\eta)} \frac{dx}{\sqrt{2\pi}} = \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} \frac{dy}{\sqrt{2\pi(1-2\eta)}}$$

$$= \frac{1}{\sqrt{1-2\eta}}.$$

Hence,

$$E\left[e^{\xi R^2(t)}\right] = \frac{1}{1-2\xi t}.$$

3. (Black-Scholes model) A derivative of European type pays the amount

$$Y = 1 + S(T) \ln S(T)$$

at time of maturity T. (a) Find $\Pi_Y(t)$. (b) Find a hedging portfolio of the derivative at time t.

Solution. (a) If $s = S(t)$, $\tau = T - t$, and $G \in N(0,1)$, then

$$\Pi_Y(t) = e^{-r\tau} E\left[1 + se^{(r-\frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau}G}\left\{\ln s + (r - \frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau}G\right\}\right]$$

$$= e^{-r\tau} + s\left\{\ln s + (r - \frac{\sigma^2}{2})\tau\right\} e^{-\frac{\sigma^2}{2}\tau} E\left[e^{\sigma\sqrt{\tau}G}\right] + s\sigma\sqrt{\tau}E\left[Ge^{-\frac{\sigma^2}{2}\tau + \sigma\sqrt{\tau}G}\right]$$

$$= e^{-r\tau} + s\left\{\ln s + (r - \frac{\sigma^2}{2})\tau\right\} + s\sigma\sqrt{\tau}\int_{-\infty}^{\infty} xe^{-\frac{\sigma^2}{2}\tau + \sigma\sqrt{\tau}x - \frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}}$$

$$= e^{-r\tau} + s\left\{\ln s + (r - \frac{\sigma^2}{2})\tau\right\} + s\sigma\sqrt{\tau}\int_{-\infty}^{\infty} xe^{-\frac{(x-\sigma\sqrt{\tau})^2}{2}} \frac{dx}{\sqrt{2\pi}} =$$

$$= e^{-r\tau} + s\left\{\ln s + (r - \frac{\sigma^2}{2})\tau\right\} + s\sigma\sqrt{\tau}\int_{-\infty}^{\infty} (y + \sigma\sqrt{\tau})e^{-\frac{y^2}{2}} \frac{dx}{\sqrt{2\pi}}$$

$$= e^{-r\tau} + s\left\{\ln s + (r - \frac{\sigma^2}{2})\tau\right\} + s\sigma^2\tau$$

$$= e^{-r\tau} + s\ln s + s(r + \frac{\sigma^2}{2})\tau$$

$$= e^{-r\tau} + S(t)\ln S(t) + S(t)(r + \frac{\sigma^2}{2})\tau$$

(b) A portfolio with

$$h_S(t) = \left(\frac{\partial}{\partial s}\left\{e^{-r\tau} + s\ln s + s(r + \frac{\sigma^2}{2})\tau\right\}\right)_{|s=S(t)}$$

$$= 1 + (r + \frac{\sigma^2}{2})\tau + \ln S(t)$$

units of the stock and

$$h_B(t)$$

$$= (e^{-r\tau} + S(t)\ln S(t) + S(t)(r + \frac{\sigma^2}{2})\tau - S(t)(1 + (r + \frac{\sigma^2}{2})\tau + \ln S(t)))/B(t)$$

$$= (e^{-r\tau} - S(t))/B(t)$$

units of the bond is a hedging portfolio at time t.

4. (Dominance Principle) State and prove the Put-Call Parity relation.

5. (Dominance Principle) Suppose $t_0 < t^* < T$ and let D be a positive number, which is known at time t_0. Now consider an American put with strike K and time of maturity T, where the underlying stock pays the dividend D at time t^* and

$$D \geq K(e^{r(t^*-t_0)} - 1).$$

Prove that it is not optimal to exercise the put in the time interval $]t_0, t^*[$.