1. (Black-Scholes model) A derivative of European type pays the amount

\[ Y = S(T) + \frac{1}{S(T)} \]

at time of maturity \( T \). Find \( \Pi_Y(t) \) for all \( 0 \leq t < T \).

Solution. We have

\[ \Pi_Y(t) = \Pi_{S(T)}(t) + \Pi_{\frac{1}{S(T)}}(t). \]

Here, if \( \tau = T - t \), \( s = S(t) \), and \( G \in \mathcal{N}(0,1) \),

\[ \Pi_{S(T)}(t) = e^{-r\tau} E \left[ s e^{(r - \frac{\sigma^2}{2})\tau + \sigma \sqrt{\tau} G} \right] = s e^{-\frac{\sigma^2}{2} \tau} e^{\frac{\sigma^2}{2} \tau} = s. \]

Moreover,

\[ \Pi_{\frac{1}{S(T)}}(t) = e^{-r\tau} E \left[ \frac{1}{s e^{(r - \frac{\sigma^2}{2})\tau + \sigma \sqrt{\tau} G}} \right] = e^{-r\tau} \frac{e^{-(r - \frac{\sigma^2}{2})\tau}}{s} E \left[ e^{\sigma \sqrt{\tau} G} \right] = e^{-(2r - \frac{\sigma^2}{2})\tau} \frac{1}{s e^{(\sigma^2 - 2r)\tau}}. \]

and it follows that

\[ \Pi_Y(t) = S(t) + \frac{1}{S(t)} e^{(\sigma^2 - 2r)\tau}. \]
2. (Binomial model) Suppose \( d = -u \) and \( e^r = \frac{1}{2}(e^u + e^d) \). A financial derivative of European type has the maturity date \( T = 4 \) and payoff \( Y = f(X_1 + X_2 + X_3 + X_4) \), where \( f(x) = 1 \) if \( x \in \{4u, 0, -4u\} \) and \( f(x) = -1 \) if \( x \in \{2u, -2u\} \). Show that \( \Pi_Y(0) = 0 \).

Solution. It follows that \( d < r < u \) and
\[
q_u = \frac{e^r - e^d}{e^u - e^d} = \frac{e^u - e^r}{e^u - e^d} = q_d.
\]
Hence \( q_u = q_d = \frac{1}{2} \). Furthermore,
\[
\Pi_Y(0) = e^{-4r} \sum_{k=0}^{4} \binom{4}{k} q_u^k q_d^{4-k} f(ku + (4-k)d)
\]
\[
= e^{-4r} \sum_{k=0}^{4} \binom{4}{k} q_u^k q_d^{4-k} f((2k-4)u)
\]
\[
= e^{-4r} \left( \frac{1}{2} \right)^4 (1 - 4 + 6 - 4 + 1) = 0.
\]

3. (Black-Scholes model) Suppose \( T > 0 \), \( N \in \mathbb{N}_+ \), \( h = \frac{T}{N} \), and \( t_n = nh \), \( n = 0, ..., N \), and consider a derivative of European type paying the amount \( Y = \sum_{n=0}^{N-1} (\ln \frac{S(t_{n+1})}{S(t_n)})^2 \) at time of maturity \( T \). Find \( \Pi_Y(0) \).

Solution. First consider a derivative paying the amount \( Y_n = (\ln \frac{S(t_{n+1})}{S(t_n)})^2 \) at time \( T \). Since \( Y_n \) is known at time \( t_{n+1} \), \( \Pi_{Y_n}(t_{n+1}) = Y_ne^{-r(T-t_{n+1})} \). Note that
\[
S(t_{n+1}) = S(t_n)e^{(\sigma^2/2)h + \sigma(W(t_{n+1}) - W(t_n))}
\]
where \( W(t_{n+1}) - W(t_n) \in \mathcal{N}(0, h) \). Thus, if \( G \in \mathcal{N}(0, 1) \),
\[
\Pi_{Y_n}(t_n) = e^{-rh} E \left[ e^{-r(T-t_{n+1})} \left\{ (r - \frac{\sigma^2}{2})h + \sigma\sqrt{h}G \right\}^2 \right]
\]
\[= e^{-r(T-t_n)} \left\{ \left( r - \frac{\sigma^2}{2} \right) T^2 + \sigma^2 h \right\} \]

and since the expression for \( \Pi_{Y_n}(t_n) \) is known at time 0,

\[\Pi_{Y_n}(0) = e^{-t_n h} e^{-r(T-t_n)} \left\{ \left( r - \frac{\sigma^2}{2} \right) T^2 + \sigma^2 h \right\} \]

\[= e^{-rT} \left\{ \left( r - \frac{\sigma^2}{2} \right) T^2 + \sigma^2 h \right\} .\]

Now it follows that

\[\Pi_Y(0) = \sum_{n=0}^{N-1} \Pi_{Y_n}(0) = N e^{-rT} \left\{ \left( r - \frac{\sigma^2}{2} \right) T^2 + \sigma^2 h \right\} \]

\[= T e^{-rT} \left\{ \sigma^2 + h \left( r - \frac{\sigma^2}{2} \right) \right\} .\]

4. Derive the delta of a European call in the Black-Scholes model. Recall that the call price equals \( s \Phi(d_1) - K e^{-r\tau} \Phi(d_2) \), where \( s = S(t), \tau = T-t > 0 \), and

\[d_1 = \frac{\ln s + (r + \frac{1}{2}) \tau}{\sigma \sqrt{\tau}} = d_2 + \sigma \sqrt{\tau} .\]

5. Consider the binomial model in one period and assume \( d < r < u \). A derivative pays the amount \( Y = f(X) \) at time 1. Find a portfolio which replicates the derivative at time 0.