Notes
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1 Definitions

Definition 1.1
At time $t < T$ is called an optimal exercise time for the American put with value $\hat{P}(t, S(t), K, T)$ if
\[ \hat{P}(t, S(t), K, T) = (K - S(t))_. \]

Definition 2.2
A portfolio process $\{h_S(t), h_B(t)\}_{t \in \mathcal{I}}$ invested in a binomial market is said to be self-financing if
\[ h_S(t)S(t-1) + h_B(t)B(t-1) = h_S(t-1)S(t-1) + h_B(t-1)B(t-1) \]
holds for all $t \in \mathcal{I}$.

Definition 2.3
A portfolio process $\{h_S(t), h_B(t)\}_{t \in \mathcal{I}}$ invested in a binomial market is called an arbitrage portfolio if its value $V(t)$ satisfies
\begin{itemize}
  \item $V(0) = 0$,
  \item $V(N, x) \geq 0 \forall x \in \{u, d\}^N$,
  \item There exists $y \in \{u, d\}^N$ such that $V(N, y) > 0$.
\end{itemize}

Definition 3.1
A portfolio process $\{h_S(t), h_B(t)\}_{t \in \mathcal{I}}$ is called predictable if there exists $N$ functions $H_1, ..., H_N$ such that $H_t : (0, \infty)^t \to \mathbb{R}^2$ and
\[ (h_S(t), h_B(t)) = H_t(S_0, ..., S(t-1)), \quad t \in \mathcal{I}. \]
Definition 3.2

A **hedging** portfolio for a European derivative with pay-off \( Y = g(S(N)) \) at expiration date \( T = N \) is a portfolio process \( \{(h_S(t), h_B(t))\}_{t \in \mathcal{I}} \) invested in the underlying stock and risk-free asset such that its value \( V(t) \) satisfies \( V(N) = Y \); the latter equality must be satisfied for all possible paths of the price of the underlying stock, i.e., \( V(N,x) = Y(x) \forall x \in \{u,d\}^N \).

Definition 3.3

The binomial (fair) price of a European derivative with pay-off \( Y \) and maturity \( N \) is given by

\[
\Pi_Y(t) := e^{-r(N-T)} \sum_{(x_{t+1},\ldots,x_N) \in \{u,d\}^{N-t}} q_{x_{t+1}} \cdots q_{x_N} Y(x_1,\ldots,x_N).
\]

Definition 4.1

A portfolio process \( \{h_S(t), h_B(t)\}_{t \in \mathcal{I}} \) is said to be hedging an American derivative with intrinsic value \( Y(t) \) if

\[
V(N) = Y(N), \quad V(t) \geq Y(t) \forall t = 0,\ldots,N-1,
\]

where \( V(t) = h_S S(t) + h_B B(t) \) is the value of the portfolio process at time \( t \).

Definition 4.2

The binomial (fair) price \( \hat{\Pi}_Y(t) \) of a standard American derivative with pay-off \( Y(t) = g(S(t)) \) at time \( t \in \{0,1,\ldots,N\} \) is defined by the recurrence formula

\[
\hat{\Pi}_Y(N) = Y(N)
\]

\[
\hat{\Pi}_Y(t) = \max(Y(t), e^{-r} (q_u \hat{\Pi}_Y(t+1) + q_d \hat{\Pi}_Y(t+1)))
\]

Definition 4.3

A replicating portfolio process for an American derivative with intrinsic value \( Y(t) \) is a portfolio process that satisfies \( V(t) = \hat{\Pi}_Y(t) \), for all \( t \in \{0,\ldots,N\} \) (and for all possible paths of the stock price).

Definition 4.4

A portfolio process \( \{h_S(t), h_B(t)\}_{t \in \mathcal{I}} \) is said to generate cash flow \( C(t-1), t \in \mathcal{I} \), if

\[
h_S(t) S(t-1) + h_B(t) B(t-1) = h_S(t-1) S(t-1) + h_B(t-1) B(t-1) - C(t-1), t \in \mathcal{I},
\]

or, equivalently

\[
V(t) - V(t-1) = h_S(t) (S(t) - S(t-1)) + h_B(t) (B(t) - B(t-1)) - C(t-1).
\]

Definition 5.4

Two events \( A \) and \( B \) are said to be independent if \( \mathbb{P}(A \cap B) = \mathbb{P}(A) \mathbb{P}(B) \).
Definition 5.15
A discrete stochastic process \( \{X_1, X_2, \ldots \} \) on the finite probability space \((\Omega, \mathbb{P})\) is called a martingale if
\[
\mathbb{E}[X_{i+1} | X_1, X_2, \ldots, X_i] = X_i \quad \forall i \geq 1.
\]

Definition 5.19
Let \( \{W(t)\}_{t \in [0,T]} \) be a Brownian motion, \( \alpha \in \mathbb{R} \), and \( \sigma > 0 \). The positive stochastic process \( \{S(t)\}_{t \in [0,T]} \)
\[
S(t) = S(0)e^{\alpha t + \sigma W(t)},
\]
is called a geometric Brownian motion.

Definition 6.1
Consider a European derivative with pay-off \( Y = g(S(T)) \) at the maturity \( T > 0 \). Assume that the price of the underlying stock is given by the geometric brownian motion \( S(t) = S(0)e^{\alpha t + \sigma W(t)} \). The Black-Scholes price \( \Pi_Y(t) \) of the derivative at time \( t \in [0,T] \) is
\[
\Pi_Y(t) = v(t, S(t)) \quad \text{where}
\]
\[
v(t, x) = \frac{e^{-rt}}{\sqrt{2\pi}} \int_{\mathbb{R}} g(xe^{(r-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}y})e^{-\frac{y^2}{2\tau}} dy, \quad \tau = T - t.
\]

2 Theorems

Theorem 1.1
Let \( C(t, S(t), T, K) \) denote the price of a European call, and let \( P(t, S(t), T, K) \) be the price of the corresponding European put. Assume that there exists a risk-free asset in the money market with constant interest rate \( r \). If the dominance principle holds, then for all \( t < T \),

1. The put-call parity holds
\[
S(t) - C(t, S(t), T, K) = Ke^{-r(T-t)} - P(t, S(t), T, K).
\]

2. If \( r \geq 0 \) then \( C(t, S(t), T, K) \geq (S(t) - K)_+ \); the strict inequality holds for \( r > 0 \).

3. If \( r \geq 0 \), the map \( T \mapsto C(t, S(t), T, K) \) is non-decreasing.

4. The maps \( K \mapsto C(t, S(t), T, K) \) and \( K \mapsto P(t, S(t), T, K) \) are convex.

Proof

1. Consider a portfolio \( \mathcal{A} \) which is long one share of the stock and one share of the put option, and short of the call and \( K/B(T) \) shares of the risk-free asset. The value of this portfolio at maturity is
\[
V_\mathcal{A}(T) = S(T) + (K - S(T))_+ - (S(T) - K)_+ - \frac{K}{B(T)} B(T) = 0.
\]
Hence by the dominance principle \( V_A \geq 0 \) for \( t < T \), that is
\[
S(t) + P(t, S(t), K, T) - C(t, S(t), K, T) - Ke^{-r(T-t)} \geq 0.
\]
Now consider the portfolio \(-A\) with the opposite position on each asset. Again we have \( V_{-A}(T) = 0 \) and thus \( V_{-A}(t) = -V_A(t) \geq 0 \) for \( t < T \). Hence
\[
S(t) + P(t, S(t), K, T) - C(t, S(t), K, T) - Ke^{-r(T-t)} \leq 0.
\]
Thus the left hand side in the previous two inequalities must be zero, which gives the put-call parity.

2. We can assume \( S(t) \geq K \) otherwise it’s trivial. By the put-call parity, using that \( P(t, S(t), K, T) \geq 0 \),
\[
C(t, S(t), K, T) = S(t) - Ke^{-r(T-t)} + P(t, S(t), K, T) \geq S(t) - Ke^{-r(T-t)};
\]
the right hand side equals \( S(t) - K \) for \( r = 0 \) and is strictly greater than this quantity for \( r > 0 \). As \( S(t) - K = (S(t) - K)_+ \) for \( S(t) \geq K \), the claim follows.

3. Consider a portfolio \( A \) which is long one call with maturity \( T_2 \) and strike \( K \) and short one call with maturity \( T_1 \) and strike \( K \), where \( T_2 > T_1 \geq t \). By claim 2 we have
\[
C(T_1, S(T_1), K, T_2) \geq (S(T_1) - K)_+ = C(t, S(T_1), K, T_1),
\]
i.e. \( V_A(T_1) \geq 0 \) for \( t < T_1 \). Hence \( V_A(t) \geq 0 \) i.e. \( C(t, S(t), K, T_2) \geq C(t, S(t), K, T_1) \), which is the claim.

4. We prove the statement for call options, the argument for put options being the same. Let \( K_0, K_1 > 0 \) and \( 0 < \lambda < 1 \) be given. Consider a portfolio \( A \) which is short one share with strike \( \lambda K_0 + (1 - \lambda)K_1 \) and maturity \( T \), long \( \lambda \) shares of a call with strike \( K_1 \) and maturity \( T \), long \( (1 - \lambda) \) shares of a call with strike \( K_0 \) and maturity \( T \). The value of this portfolio at maturity is
\[
V_A(T) = -(S(T) - (\lambda K_1 + (1 - \lambda)K_0))_+ + \lambda(S(T) - K_1)_+ + (1 - \lambda)(S(T) - K_0)_+.
\]
The convexity of the function \( f(x) = (S(T) - x)_+ \) gives \( V_A(T) \geq 0 \), and so \( V_A(t) \geq 0 \) by the dominance principle. The latter inequality is
\[
C(t, S(t), \lambda K_1 + (1 - \lambda)K_0, T) \leq \lambda C(t, S(t), K_1, T) + (1 - \lambda)C(t, S(t), K_0, T),
\]
which is the claim for call options.

**Theorem 2.1**

Let \( \{h_S(t), h_B(t)\}_{t \in \mathcal{I}} \) be a self-financing portfolio process with value \( V(N) \) at time \( t = N \). Define
\[
q_u = \frac{e^r - e^d}{e^u - e^d}, \quad q_d = 1 - q_u.
\]
Then for \( t = 0, \ldots, N - 1 \), \( V(t) \) is given by
\[
V(t) = e^{-r(N-t)} \sum_{(x_{t+1}, \ldots, x_N) \in \{u,d\}^{N-t}} q_{x_{t+1}} \cdots q_{x_N} V(N, x).
\]
In particular we have the initial value
\[ V(0) = e^{-rN} \sum_{x \in \{u,d\}} q_u^{N_u(x)} q_d^{N_d(x)} V(N,x). \]

Moreover the portfolio satisfies the recurrence formula
\[ V(t - 1) = e^{-r}[q_u V_u(t) + q_d V_d(t)], \quad t \in \mathcal{I}, \]
where
\[ V_u(t) = h_S(t) S(t - 1) e^u + h_B(t) B(t - 1) e^r, \]
\[ V_d(t) = h_S(t) S(t - 1) e^d + h_B(t) B(t - 1) e^d. \]

**Proof**

We prove it by induction for \( t = 0, ..., N - 1 \).

**Step 1**

We begin with \( t = N - 1 \). Then
\[ V(N - 1) = e^{-r}[q_u V(N, (x_1, ..., x_{N-1}, u)) + q_d V(N, (x_1, ..., x_{N-1}, d))] . \tag{1} \]

Here, we have
\[ V(N, (x_1, ..., x_{N-1}, u)) = h_S(N) S(N - 1) e^u + h_B(N) B(N - 1) e^r, \]
similarly for \( V(N, (x_1, ..., x_{N-1}, d)) \) but \( u \) replaced with \( d \), which follows by the definition of portfolio value. Thus \( V(N - 1) \) is equal to
\[
V(N - 1) \\
= e^{-r}[q_u (h_S(N) S(N - 1) e^u + h_B(N) B(N - 1) e^r) \\
\quad + q_d (h_S(N) S(N - 1) e^d + h_B(N) B(N - 1) e^r)] \\
= e^{-r}[h_S(N) S(N - 1) e^r + h_B(N) B(N - 1) e^r] \\
= h_S(N) S(N - 1) + h_B(N) B(N - 1),
\]

since \( e^u q_u + e^d q_d = e^r \) and \( q_u + q_d = 1 \). This proves the claim for \( t = N - 1 \), by the definition of self-financing portfolios.

**Step 2**

Now assume this is true at \( t + 1 \) i.e.
\[ V(t + 1) = e^{-r(N-t-1)} \sum_{(x_{t+2}, ..., x_N) \in \{u,d\}^{N-t-1}} q_{x_{t+2}} \cdots q_{x_N} V(N, x). \tag{2} \]
Step 3
We now prove it at time $t$. Let 
\[ 
V^u(t + 1) := h_S(t + 1)S(t)e^u + h_B(t + 1)B(t)e^r \quad \text{assuming } x_{t+1} = u, \\
V^d(t + 1) := h_S(t + 1)S(t)e^d + h_B(t + 1)B(t)e^r \quad \text{assuming } x_{t+1} = d. 
\]

This gives us 
\[ 
e^{-r}[q_u V^u(t + 1) + q_d V^d(t + 1)] = h_S(t + 1)S(t) + h_B(t + 1)B(t). 
\]

By the self financing property we have 
\[ 
e^{-r}[q_u V^u(t + 1) + q_d V^d(t + 1)] = V(t), 
\]
which proves that $V$ satisfies the recurrence formula. Moreover, with the induction hypothesis we have 
\[ 
V^u(t + 1) = e^{-r(N-t-1)} \sum_{(x_{t+2}, \ldots, x_N) \in \{u,d\}^{N-t-1}} q_{x_{t+2}} \cdots q_{x_N} V(N, x_1, \ldots, x_{t+1}, u, x_{t+2}, \ldots, x_N), \\
V^d(t + 1) = e^{-r(N-t-1)} \sum_{(x_{t+2}, \ldots, x_N) \in \{u,d\}^{N-t-1}} q_{x_{t+2}} \cdots q_{x_N} V(N, x_1, \ldots, x_{t+1}, d, x_{t+2}, \ldots, x_N), 
\]
using these, with equation (2), we obtain 
\[ 
V(t) = e^{-r(N-T)} \sum_{(x_{t+1}, \ldots, x_N) \in \{u,d\}^{N-t}} q_{x_{t+1}} \cdots q_{x_N} V(N, x). 
\]

**Theorem 2.2**
The binomial market is arbitrage free iff $r \in (d, u)$.

**Proof**
The proof is divided into 2 steps, first we prove the claim for the 1-period model. The generalization for the multiperiod model $N > 1$ is carried out in step 2.

**Step 1**
Let the portfolio position in the 1-period model be constant, (thus be self-financing over $[0, 1]$), i.e. let 
\[ 
h_S(0) = h_S(1) = h_S, \quad h_B(0) = h_B(1) = h_B. \quad (3) 
\]
The portfolio value at $t = 0$ is 
\[ 
V(0) = h_S S_0 + h_B B_0, \quad (4) 
\]
while at time $t = 1$ it is 
\[ 
V(1) = \begin{cases} 
V(1, u) = h_S S_0 e^u + h_B B_0 e^r & \text{if stock goes up at } t = 1 \\
V(1, d) = h_S S_0 e^d + h_B B_0 e^r & \text{if stock goes down at } t = 1 
\end{cases} \quad (5) 
\]
Thus the portfolio is an arbitrage if \( V(0) = 0 \), i.e.
\[ h_S S_0 + h_B B_0 = 0, \]
if \( V(1) \geq 0 \) i.e.
\[ h_S S_0 e^u + h_B B_0 e^r \geq 0, \]
\[ h_S S_0 e^d + h_B B_0 e^r \geq 0, \]
and if at least one of the two inequalities in (7) is strict. Now assume that \((h_S, h_B)\) is an arbitrage portfolio. From (6) we have \( h_S S_0 = -h_B B_0 \), thus (7) becomes
\[ h_S S_0 (e^u - e^r) \geq 0, \]
\[ h_S S_0 (e^d - e^r) \geq 0. \]
We have \( h_S \neq 0 \) since at least one of the inequalities must be strict. Assuming \( h_S > 0 \) then we obtain from the two inequalities above that \( d \geq r \). Instead, assuming \( h_S < 0 \) we instead obtain \( r \geq u \). Hence the existence of an arbitrage portfolio implies \( r \geq u \) or \( r \leq d \), i.e. \( r \notin (d, u) \). Which proves that for \( r \in (d, u) \) there is no arbitrage portfolio for the 1-period model. Now we need to prove that \( r \in (d, u) \) is necessary for the absence of arbitrages, we construct an arbitrage portfolio when \( r \notin (d, u) \). Assume \( r \leq d \), pick \( h_S = 1 \) and \( h_B = -S_0 / B_0 \). Then \( V(0) = 0 \). Further, \( h_S S_0 e^d + h_B B_0 e^r \geq 0 \) is trivially satisfied, and since \( u > d \) we have
\[ h_S S_0 e^u + h_B B_0 e^r = S_0 (e^u - e^r) > S_0 (e^d - e^r) \geq 0. \]
This shows that one can construct an arbitrage portfolio when \( r \leq d \), a similiar argument is done for \( r \geq u \). We now continue with step 2.

**Step 2**

Again let \( r \notin (d, u) \), we’ve shown that in the 1-period model there exists an arbitrage portfolio \((h_S, h_B)\). Now by investing the whole value of the portfolio \((h_S, h_B)\) at \( t = 1 \) in the risk-free asset, we can build a self-financing arbitrage portfolio process \( \{h_S(t), h_B(t)\}_{t \in \mathbb{I}} \) for the multiperiod model. This portfolio satisfies \( V(0) = 0 \) and \( V(N, x) = V(1, x) e^{r(N-1)} \geq 0 \) along every path \( x \in \{u, d\}^N \). Moreover, \((h_S, h_B)\) is an arbitrage, therefore \( V(1, y) > 0 \) for some \( y \in \{u, d\}^N \), hence \( V(N, y) > 0 \). The constructed self-financing portfolio process \( \{h_S(t), h_B(t)\}_{t \in \mathbb{I}} \) is an arbitrage, now we have to prove the “only if” part for the multiperiod model. By Theorem 2.1
\[ V(0) = e^{-rN} \sum_{x \in \{u, d\}^N} (q_u)^{N_u(x)} (q_d)^{N_d(x)} V(N, x). \]
Now assume that the portfolio is an arbitrage. Then \( V(0) = 0 \) and \( V(N, x) \geq 0 \). We can consider only paths such that \( V(N, x) > 0 \) which exists since the portfolio is an arbitrage. But then (11) can be zero only if one of \( q_u \) or \( q_d \) is zero, or if opposite signs. Since \( u > d \) we have
\[ q_u = 0, \quad \text{resp. } q_d = 0 \Rightarrow r = d, \quad \text{resp. } u = r \]
\[ (q_u > 0, q_d < 0), \quad \text{resp. } (q_u < 0, q_d > 0) \Rightarrow u < r, \quad \text{resp. } r < d. \]
We conclude that the existence of a self-financing arbitrage portfolio entails \( r \notin (d, u) \) which completes the proof.
Theorem 3.2
Consider a standard European derivative with pay-off $Y = g(S(N))$ at the time of maturity $N$. Then the portfolio given by

$$h_S(0) = h_S(1), \quad h_B(0) = h_B(1),$$

and for $t \in \mathcal{I}$,

$$h_S(t) = \frac{1}{S(t-1)} \frac{\Pi^u_Y(t) - \Pi^d_Y(t)}{e^u - e^d},$$

$$h_B(t) = \frac{e^{-r}}{B(t-1)} \frac{e^u \Pi^u_Y(t) - e^d \Pi^u_Y(t)}{e^u - e^d},$$

is a self-financing, predictable, hedging portfolio.

Proof
We begin proving the hedging property, we have

$$V(t) = h_S(t)S(t) + h_B(t)B(t) = \frac{S(t)}{S(t-1)} \frac{\Pi^u_Y(t) - \Pi^d_Y(t)}{e^u - e^d} + \frac{e^{-r}B(t)}{B(t-1)} \frac{e^u \Pi^u_Y(t) - e^d \Pi^u_Y(t)}{e^u - e^d}. $$

Here $e^{-r}B(t)/B(t-1) = 1$ and $S(t)/S(t-1)$ is either $e^u$ or $e^d$ depending on $S(t)$. Using these two values we obtain $V^u_Y(t) = \Pi^u_Y(t)$, and $V^d_Y(t) = \Pi^d_Y(t)$, that is $V(t) = \Pi_Y(t)$ i.e. replicating the derivative. In particular, for $t = N$ we have $V(N) = \Pi_Y(N) = Y$, hence the portfolio is hedging the derivative.

Now, proving the self-financing property, we have

$$h_S(t)S(t-1) + h_B(t)B(t-1) = \frac{\Pi^u_Y(t)(1 - e^{d-r}) + \Pi^d_Y(t)(e^{u-r} - 1)}{e^u - e^d} = \Pi_Y(t-1),$$

by using the definition of $q_u, q_d$ as well as the recurrence formula. Also we already know that the portfolio is replicating the derivative, i.e. $V(t-1) = \Pi_Y(t-1)$, therefore

$$h_S(t)S(t-1) + h_B(t)B(t-1) = V(t-1).$$

Finally, the portfolio is predictable, since

$$\Pi_Y := e^{-r(N-t)} \sum_{(x_{t+1}, ..., x_N) \in \{u,d\}^{N-t}} q_{x_{t+1}} \cdots q_{x_N} g(S(t) \exp(x_{t+1} + \ldots + x_N)),$$

therefore, we have

$$\Pi^u_Y(t) := e^{-r(N-t)} \sum_{(x_{t+1}, ..., x_N) \in \{u,d\}^{N-t}} q_{x_{t+1}} \cdots q_{x_N} g(S(t-1) e^u \exp(x_{t+1} + \ldots + x_N)),$$

hence $\Pi^u_Y(t)$ is a deterministic function of $S(t-1)$ and the same property holds for $\Pi^d_Y(t)$. Thus $h_S(t), h_B(t)$ are deterministic functions of $S(t-1)$, which proves that the portfolio is predictable.
Theorem 4.1
Consider a standard American derivative with intrinsic value $Y(t)$ and let $\hat{\Pi}_Y(t)$ be its binomial fair price. Define the portfolio process $\{\hat{h}_S(t), \hat{h}_B(t)\}_{t \in \mathcal{I}}$ and the cash flow process $C(t)$ recursively as follows:

\begin{align*}
C(0) &= 0, \quad C(t-1) = \hat{\Pi}_Y(t-1) - e^{-r}[q_u\hat{\Pi}_u(t) + q_d\hat{\Pi}_d(t)], \quad t \in \{2, ..., N\} \quad (13) \\
\hat{h}_S(1) &= \hat{h}_S(0), \quad \hat{h}_B(1) = \hat{h}_B(0), \quad (14)
\end{align*}

and for $t \in \{1, ..., N\},$

\begin{align*}
\hat{h}_S(t) &= \frac{1}{S(t-1)} \frac{\hat{\Pi}_u(t) - \hat{\Pi}_d(t)}{e^u - e^d}, \quad (15) \\
\hat{h}_B(t) &= \frac{e^{-r}}{B(t-1)} \frac{e^u\hat{\Pi}_d(t) - e^d\hat{\Pi}_u(t)}{e^u - e^d}. \quad (16)
\end{align*}

Then the value of this portfolio process satisfies

\begin{align*}
V(t) &= \hat{\Pi}_Y(t) \forall t \in \{0, ..., N\}, \quad (17)
\end{align*}

and

\begin{align*}
V(t-1) &= \hat{h}_S(t)S(t-1) + \hat{h}_B(t)B(t-1) + C(t-1), \quad \forall t \in \mathcal{I}. \quad (18)
\end{align*}

Proof
By using the equations (13), (14), (15), (16) into equations (17) and (18) we obtain

\begin{align*}
V^u(t) = \hat{h}_S(t)S(t-1)e^u + \hat{h}_B(t)B(t-1)e^d = \hat{\Pi}_u(t),
\end{align*}

similiar calculations proves $V^d(t) = \hat{\Pi}_d(t)$, hence (17) holds. Also replacing equations (13), (14), (15), (16) into the right hand side of (18), the latter is equal to $\hat{\Pi}_Y(t-1)$, which we proved is equal to $V(t-1)$, hence (18) holds.

Theorem 5.3
If $r \notin (d,u)$ there is no probability measure $\mathbb{P}_p$ on the sample space $\Omega_N$ such that the discounted stock price $\{\hat{S}(t)\}_{t \in \mathcal{I}}$ is a martingale. For $r \in (d,u),$ $\{\hat{S}(t)\}_{t \in \mathcal{I}}$ is a martingale with respect to the probability measure $\mathbb{P}_q$ where

\begin{align*}
q = \frac{e^r - e^d}{e^u - e^d}.
\end{align*}

Moreover $\mathbb{P}_q$ is the only probability measure on $\Omega_N$ for which $\{\hat{S}(t)\}_{t \in \mathcal{I}}$ is a martingale.

Proof
By definition $\{\hat{S}(t)\}_{t \in \mathcal{I}}$ is a martingale if and only if

\begin{align*}
\mathbb{E}[e^{-rt}S(t)|\hat{S}(1), ..., \hat{S}(t-1)] = e^{-r(t-1)}S(t-1), \forall t \in \mathcal{I}. \quad (19)
\end{align*}
Clearly, conditioning on $\hat{S}(1), \ldots, \hat{S}(t-1)$ is the same as taking the expectation conditional to $S(1), \ldots, S(t-1)$, hence (19) is equivalent to
\[
\mathbb{E}[S(t)|S(1), \ldots, S(t-1)] = e^t S(t-1), \forall t \in \mathcal{I}.
\]
Moreover
\[
\mathbb{E}[S(t)|S(1), \ldots, S(t-1)] = \mathbb{E}[S(t-1) \frac{S(t)}{S(t-1)}|S(1), \ldots, S(t-1)],
\]
however since $S(t-1)$ is known, and because $S(t)/S(t-1)$ is either $e^u$ with probability $p$, or $e^d$ with probability $1-p$, and is independent of $S(1), \ldots, S(t-1)$, it follows that
\[
\mathbb{E}[S(t)|S(1), \ldots, S(t-1)] = S(t-1)(e^u p + e^d (1-p)).
\]
Therefore $\mathbb{E}[S(t)|S(1), \ldots, S(t-1)] = e^t S(t-1)$ holds iff $e^r = e^u p + e^d (1-p)$. The latter has a solution $p \in (0,1)$ iff $r \in (d,u)$, when it exists it is given by $p = q$.

**Theorem 5.4**

Let $\mathbb{E}_p[\cdot]$ denote the expectation in the probability measure $\mathbb{P}_p$. We have
\[
\mathbb{E}_p[S(N)] = S(0)(e^u p + e^d (1-p))^N, \quad \mathbb{E}_q[S(N)] = S(0)e^{rN}.
\]

**Proof**

We prove only the first formula because the second formula follows by the first one using that $e^u q + e^d (1 - q) = e^r$. We have
\[
\mathbb{E}_p[S(N)] = \mathbb{E}_p[S(0) \exp(X_1 + \ldots + X_N)] = S(0)\mathbb{E}_p[Y],
\]
where $Y$ is the random variable $Y = \exp(X_1 + \ldots + X_2) = \exp(u N_H(\omega) + d N_T(\omega)), \omega \in \Omega$. Now using that $N_T = N - N_H$ it follows that
\[
\mathbb{E}_p[S(N)] = S(0) \sum_{\omega \in \Omega_N} e^{u N_H + d N_T} p^{N_H} (1-p)^{N_T} = S(0) e^{dN} (1-p)^N \sum_{\omega \in \Omega_N} \left( \frac{e^u p}{e^d (1-p)} \right)^{N_H}.
\]
Now, since for $k = 0, \ldots, N$ there is $\binom{N}{k}$ sample points $\omega \in \Omega_N$ such that $N_H(\omega) = k$, we rewrite the above expression and using the binomial theorem, we obtain the following
\[
\mathbb{E}_p[S(N)] = S(0) e^{N_d} (1-p)^N \sum_{k=0}^{N} \binom{N}{k} \left( \frac{e^u p}{e^d (1-p)} \right)^k = S(0) e^{N_d} (1-p)^N \left( 1 + \frac{e^u p}{e^d (1-p)} \right)^N
\]
\[
= S(0)(e^d (1-p)e^u p)^N.
\]

**Theorem 5.10**

The density of the random variable $S(t)$ is given by
\[
f_{S(t)}(x) = \frac{\mathbb{1}_{x>0}}{x \sigma \sqrt{2\pi t}} \exp \left( -\frac{(\log x - \log S(0) - at)^2}{2 \sigma^2 t} \right),
\]
where $\mathbb{1}_{x>0}$ is the indicator function of the set $x > 0$. 


Proof
The density of $S(t)$ is given by
\[ f_S(t)(x) \frac{d}{dx} F_S(t)(x), \]
where $F_S(t)$ is the distribution of $S(t)$, i.e.,
\[ F_S(t)(x) = P(S(t) \leq x). \]
Clearly, $f_S(t) = F_S(t) = 0$ for $x \leq 0$. For $x > 0$ we use that
\[ S(t) \leq x \quad \text{if and only if} \quad W(t) \leq \frac{1}{\sigma} \left( \log \frac{x}{S(0)} - \alpha t \right) := A(x). \]
Thus
\[ P(S(t) \leq x) = P(-\infty < W(t) \leq A(x)) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{A(x)} e^{-\frac{y^2}{2t}} dy, \]
where for the second equality we used that $W(t) \in \mathcal{N}(0, t)$. Hence
\[ f_S(t)(x) = \frac{d}{dx} \left( \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{A(x)} e^{-\frac{y^2}{2t}} dy \right) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{A(x)^2}{2t}} \frac{dA(x)}{dx} \]
for $x > 0$, that is
\[ f_S(t)(x) = \frac{1}{\sigma x \sqrt{2\pi t}} \exp \left( -\frac{(\log x - \log S(0) - \alpha t)^2}{2\sigma^2 t} \right), \quad x > 0. \]

Theorem 6.2
The Black-Scholes price at time $t$ of a European call option with strike $K > 0$, maturity time $T > 0$ is given by $C(t, S(t))$ where
\[ C(t, x) = x \Phi(d_1) - Ke^{-rT} \Phi(d_2), \quad d_2 = \frac{\log \left( \frac{x}{K} \right) + \left( r - \frac{1}{2} \sigma^2 \right) T}{\sigma \sqrt{T}}, \quad d_1 = d_2 + \sigma \sqrt{T}, \quad (23) \]
where $\Phi$ is the standard normal distribution. The Black-Scholes price of the corresponding put option is given by $P(t, S(t))$ where
\[ P(t, x) = -x \Phi(-d_1) + Ke^{-rT} \Phi(-d_2). \]
Moreover the put-call parity holds
\[ C(t, S(t)) - P(t, S(t)) = S(t) - Ke^{-rT}. \quad (24) \]
Proof

We derive the price for call options, since the argument is similiar for put options. Recall the pay-off function \( g(z) = (z - K)_+ \), we have

\[
C(t, x) = e^{-rt} \sqrt{2\pi} \int_R g \left( x e^{(r-\frac{\sigma^2}{2})+\sigma \sqrt{\tau} y} - K \right) e^{-\frac{y^2}{2}} dy.
\]

Note that \( g \) is nonzero when \( x e^{(r-\frac{\sigma^2}{2})+\sigma \sqrt{\tau} y} - K \), i.e. when \( y > -d_2 \). Thus using \(-\frac{1}{2}y^2 + \sigma \sqrt{\tau} y = -\frac{1}{2}(y - \sigma \sqrt{\tau})^2 + \frac{a^2 \tau}{2} \). Thus we obtain

\[
C(t, x) = e^{-rt} \frac{1}{\sqrt{2\pi}} \left( x e^{r \frac{-\sigma^2}{2} + d_2^2} \int_{-d_2}^{\infty} e^{-\frac{1}{2}(y-\sigma \sqrt{\tau})^2} dy - K \int_{-d_2}^{\infty} e^{-\frac{y^2}{2}} dy \right) =
\]

Now consider the left integral with the change of variables \( u = y - \sigma \sqrt{\tau} \), which gives the lower integral limit \( u = -d_2 - \sigma \sqrt{\tau} = -d_1 \). Now since we have two integrals of even functions, symmetric around zero, we have

\[
C(t, x) = e^{-rt} \frac{1}{\sqrt{2\pi}} \left( x e^{r \frac{-\sigma^2}{2} + d_1^2} \int_{-d_1}^{\infty} e^{-\frac{1}{2}u^2} du - K \int_{-d_2}^{\infty} e^{-\frac{y^2}{2}} dy \right) =
\]

Finally, the put-call parity follows since

\[
C(t, x) - P(t, x) = x \Phi(d_1) - Ke^{-rt} \Phi(d_2) - (-x \Phi(-d_1) + Ke^{-rt} \Phi(-d_2)) =
\]

since \( \Phi(u) + \Phi(-u) = 1 \). Thus the claims follows.